FLUID DYNAMICS AND AERODYNAMICS

MCE 412

INTRODUCTION TO FLUID DYNAMICS

Fluid Mechanics is that branch of applied science that studies the action of forces on fluids and the effects these forces can produce using fluids.

Like mechanics of solids, fluid mechanics is logically divided into static and dynamics. In statics, forces acting on fluids at rest were studied. Fluid statics is also called hydrostatics, when we consider only liquid fluids. In dynamics, forces acting on the moving fluid are studied.

In the study of statics, fluid weight (acting vertically downward) and static thrust (acting on the faces of the fluid) were the significant properties of interest; but when a fluid begins to move; it obeys the same basic laws of motion as a solid body. That means a force must have been applied to generate the resultant acceleration, and the only way of applying a force to the element of fluid is by modifying the pressures extended on the fluid by the surrounding liquids.

Fluid dynamics is divided into four divisions

- Hydro-kinematics: deals separately with the motion of liquid fluids (without considering the forces responsible for the motion)
- Hydro-kinetics-deals separately with the forces responsible for the motion of liquid fluids
- Gas dynamics-deals with the motion of compressible fluids (gases and vapours)
- Aerodynamics- deals with the interaction of the atmosphere with a solid body in motion.

At the level of studying fluid dynamics, the subject should be structured in such a way; that both the motion of liquids and gases are treated simultaneously, using the same basic method of analysis.

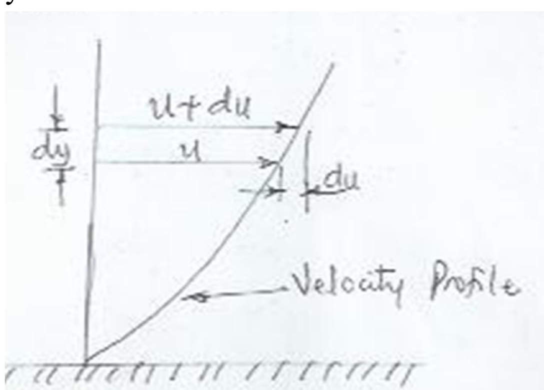
For a fundamental treatment, it is necessary to examine the forces and energies involved in fluid motion. Thus the significant properties of interest are velocity and acceleration.

The branch of fluid dynamics where velocity and acceleration are the significant properties of interest is called “kinematics of fluid flow”. The fundamental equations for studying the kinematics of fluid flow are developed from the basic laws governing the motion of solid body. These include:

(1) The conservation of mass; from which the continuity equation is developed
(2) The conservation of energy from which the energy equations are derived
(3) The conservation of momentum; from which equations evaluating dynamics forces exerted by the flowing fluids are established.

Properties of Fluids

- ***Density or mass Density***: Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume. Thus mass per unit volume of a fluid is called density. It is denoted by symbol e(rho). The unit of mass density in SI in kg/m^3 to its volume. It is denoted by the symbol w.
- ***Specific volume***: Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume. It is expressed as m^3/kg. It is commonly applied to gasses.
- ***Specific gravity***: Specific gravity is defined as the ratio of the weight density (or density) of a fluid to the weight density (or density) of a standard fluid. It is a dimensionless quantity and is denoted by the symbol S.
- ***Viscosity:*** Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluids. When two layers of a fluid at distance 'dy' apart, moves over each other at different velocities, say u + du as shown below, the viscosity together with relative velocity causes a shear stress acting between the fluid layers.



$\longrightarrow$ U

Velocity variation near a solid boundary

The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer. This shear stress is proportional to the rate of change of velocity with respect to y. it is denoted by symbol τ (Tau)

Mathematically, $\tau \propto \frac{du}{dy}$

Or $\tau = \mu \frac{dy}{dy}$

Types of Fluids

Fluids may be classified into the following types:

- Ideal fluid: A fluid, which is incompressible and having no viscosity, is known as an ideal fluid. An ideal fluid is only an imaginary fluid . All existing fluids have some viscosity.
- Real fluid: A fluid which possesses viscosity is known as real fluid. All the fluids, in actual practice are real fluids.
- Newtonian Fluids: A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient) is known as Newtonian fluid.
- Non-Newtonian fluid: A real fluid in which the shear stress is not proportional to the rate of shear strain (or velocity gradient) is known as a Non-Newtonian fluid.
- Ideal plastic fluid. A fluid in which shear stress is more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient is known as an ideal plastic fluid.

Methods for studying or describing fluid flow patterns/movement of flow

There are two methods for studying the movement of flow. One is a method which follows arbitrary particle with its kaleidoscopic changes in velocity and acceleration. This is called the Lagrangian method. The other is a method by which, rather than following any particular fluid particle, changes in velocity and pressure are studied at fixed positions in space x,y,z and at time t. This method is called the Eulerian method. Nowadays the latter method is more common and effective in most cases.

The Lagrangian method

The langrangian method is best illustrated using the solid body mechanics. In the solid body mechanics, we are used to describing the motion of a body in terms of its position, versus time.

The langrangian method is commonly used for studying the kinematics of solids where it is convenient to identify a discrete, particle e.g. Centre of mass of spring-mass system and to determine the subsequent history of its movement in time. The langrangian method when applied to a fluid body as a continuum of particle the method becomes extremely cumbersome. Hence, it is necessary to adopt a different method.

In a fluid body it is necessary to observe the motion of the fluid particle as they pass a given location in the flow field. Unlike a solid body, when a fluid body moves from one position to the next, it is usually deforms continuously. Therefore, in order to describe completely the motion of a fluid body, it is necessary to account for its deformation as well as its translation and rotation.

In addition, it is often necessary to determine the velocity and pressure distribution about a fluid body with given size and shape. Information about the flow is required at specified location in the flow field. The method of analysis that seek to analyse the motion of fluid particles (as a body) as the fluid body pass a given location is called Eulerian method or the control volume method.

The Eulerian Method

The Eulerian method, generally called the "Control Volume" method enables one to fix attention at discrete point without regards to the identity of the individual particles of fluid occupying these points at a givens instant. In using the Eulerian method, the observer notes the flow characteristic in the vicinity of a fixed point as particles pass by.

The description of the entire flow field is essentially an instantaneous picture of the velocities and accelerations of every particles as a body.

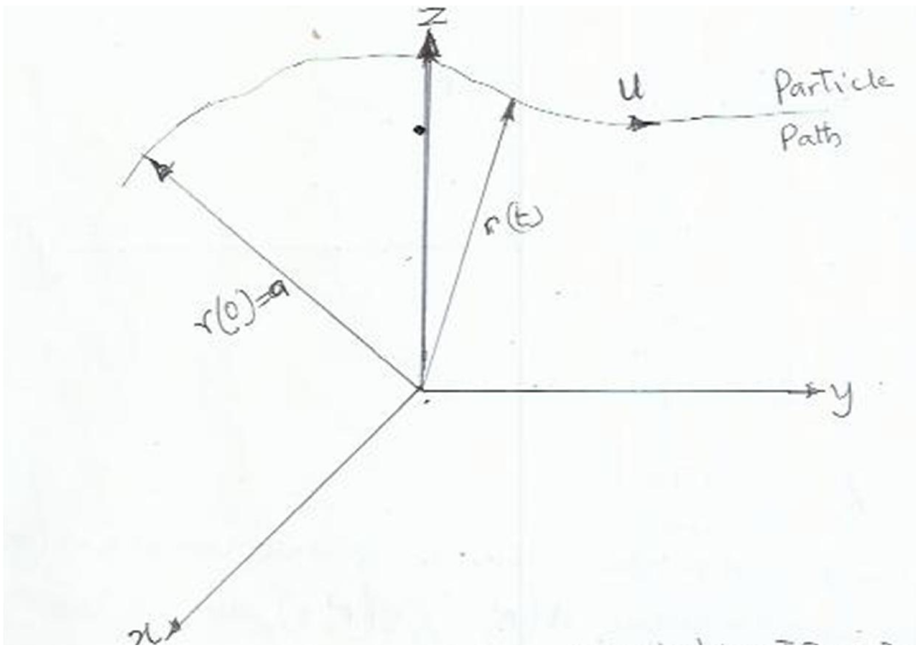
The main difference between the Lagrangian and the Eulerian method lie in the fact:

(1) In the langrangian approach, the coordinates of the particles are represented as functions of time and hence they are dependent variables.
(2) In the Eulerian approach, the particle velocities as a body at various points are given as functions of time and hence they are independent variables.

To apply the Eulerian method, first, a relationship between the two basic methods (the langrangian and the Eulerian methods) would have to be developed.

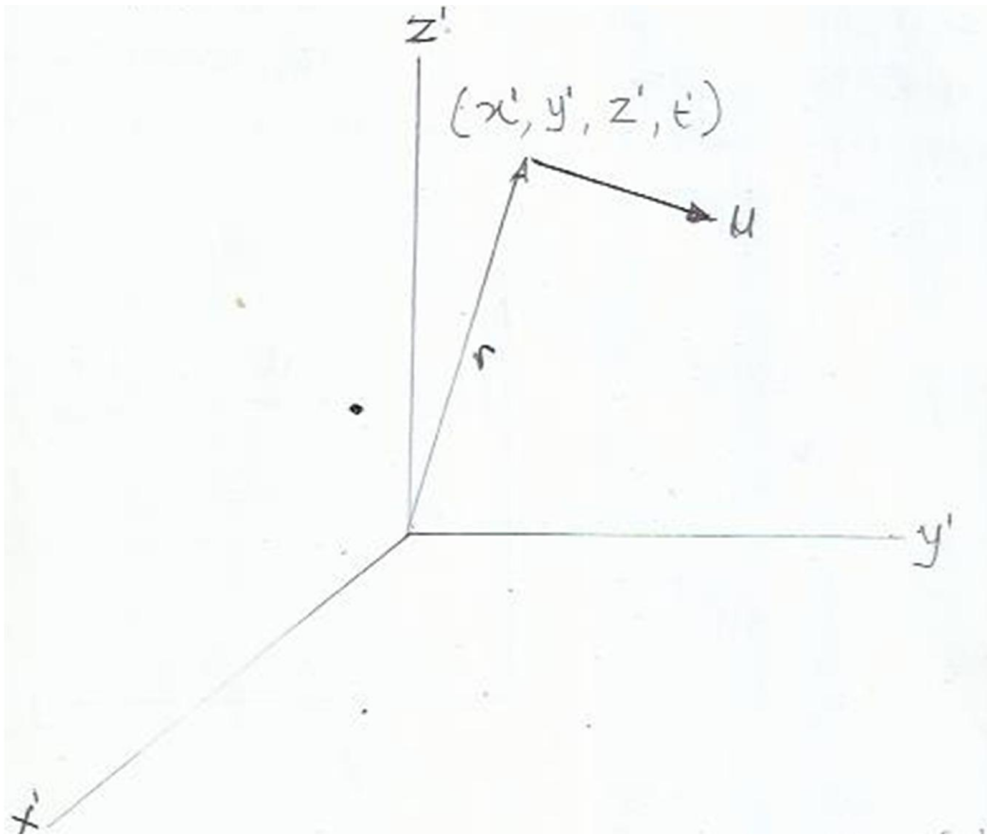
Second, since the laws governing the motion of solid body also apply to fluid body, these laws have to be adopted to the motion of fluid.

In Langrangian description, one essentially follows the history of individual fluid particles (see the figure below). Consequently, the two independent variables are taken as time and a label for fluid particles. The label can conveniently be taken as the position vector a of the particle at some reference time t = 0. In this description, any flow variable F is expressed as F(a,t). In particular , the position vector is written as r = r(a,t), which represents the location at t of a particle whose position was a at t = 0



Particle – Langrangian description. Independent variables (a,t); dependent variables: r (a,t), u= $(\partial r / \partial t)_a$ $e = \rho(a,t)$ and so on.

In the Eulerian description, one concentrates on what happens at a spatial point r^1, so that the independent variables are taken as r^1 and t^1. (Here the primes are meant to distinguish Langrangian dependent variables from Eulerian independent variables). Flow variables are written, for example as $F(r^1,t^1)$



Field-Eulerian description. Independent variables (x^1, y^1, z^1, t^1) dependent variable : $u(r^1, t^1), \rho(r^1, t)$ and so on.

The velocity and acceleration of a fluid particle in the Lagrangian description are simply the partial time derivatives as the particle identity is kept constant during the differentiation.

$u = \partial r / \partial t$, acceleration $a = \partial u / \partial t = \partial^2 r / \partial t^2$

Basic Scientific Laws used in the Analysis of fluid flow

As noted earlier in the introduction, the basic laws are related to the conservation of: mass, momentum and energy.

(i) **Law of conservation of Mass**: This law when applied to a control volume states that the net mass flow through the volume will equal the mass stored or removed from the volume. Under conditions of steady flow this will mean that the mass leaving the control volume should be equal to the mass entering the volume. The determination of flow velocity for a specified mass flow rate and flow area is based on the continuity equation derived on the basis of this law.

(ii) **Newton's law of motion**: These are basic to any force analysis under various conditions of flow. The resultant force is calculated using the condition that it equals the rate of change of momentum. The reaction on surfaces are calculated on the basis of these laws. Momentum equation for flow is derived based on these laws.

(iii) **Law of Conservation of Energy:** Considering a control volume, the law can be stated as "the energy flow into the volume will equal the energy flow out of the volume under "steady conditions". This also leads to the situation that the total energy of a fluid element in a steady flow field is conserved. This is the basis for the derivation of Euler and Bernoulli equations for fluid flow.

Note: A control volume by definition is a volume fixed with respect to a coordinate system in a flow field.

Fundamental Definition of Fluid Particles

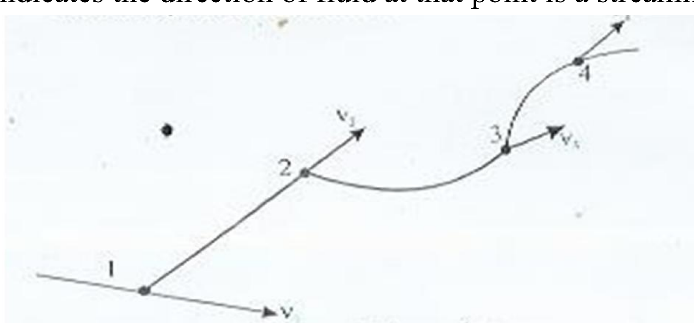
To describe a fluid particle in a flow, there must exist a flow field. This is a region in which the flow is defined in terms of space and time co-ordinate.

Generally, a fluid consists of a large number of individual particles moving in the general direction of flow, but usually not parallel to each other.

The velocity of any particle is a vector quantity having magnitude and direction which may vary from movement to movement. Thus, we have the following description of fluid particle and the flow fluid.

(i) Path Line: This is the path followed by a particle in the flow fluid. At any given time, the positions of the successive particles can be joined by a curve.

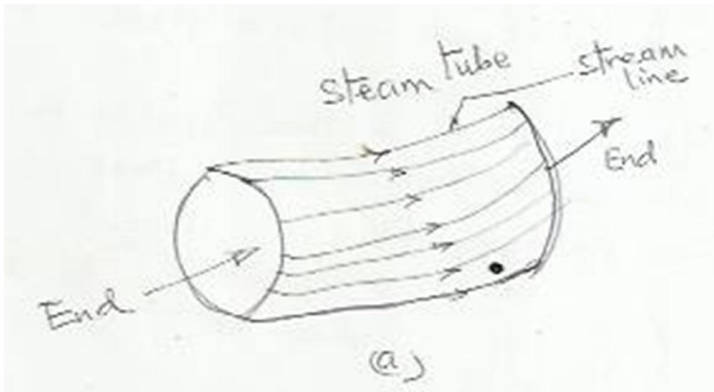
(ii) Streamline: This is a curve joining the positions of successive particles in a flow field. The curve is tangential to the direction of motion of the particle at that instant, and it is ordinary in three dimensions. In other words, the curve where the tangent at each point indicates the direction of fluid at that point is a streamline.



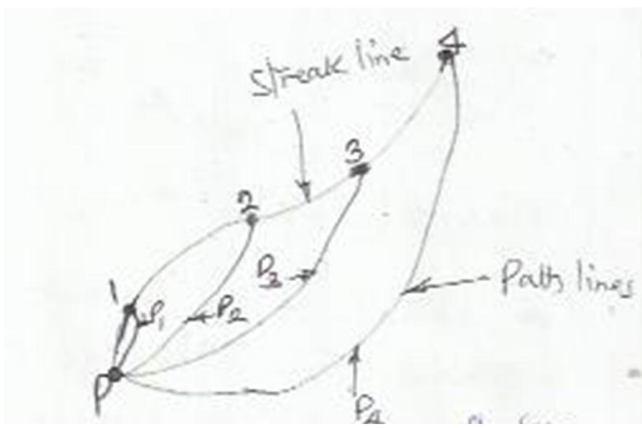
A stream line

In a flow field, the velocity of a particle at any point on a stream line is tangential to it, so there can be no flow across a stream line. A stream line therefore shows the direction of the velocity vector at any instant.

(iii) Stream tube: This is an imaginary flow passage formed by a number of adjacent stream lines. This tube is forced by drawing through every point on the circumference of a small area in the flow field.



Stream tubes



Path lines and Streak lines

If the cross-sectional area of the stream tube is small enough for the velocity to be considered constant, the flow passage is called a "stream filament".

(iv) Streak line: This is a line of fluid particles, all of which passed through the same point in the flow field at a previous time.

In experimental work, streak lines are obtained by injecting dye, smoke or particles into the moving fluid and observing the subsequent flow pattern.

Nature or types of fluid flow

There are many ways to classify fluid flow problems, the following are some general categories:

- **Steady and Unsteady flow**

Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc at a point do not change with time. Thus, for steady flow,

$$\left(\frac{\partial v}{\partial t}\right)_{x_o,y_o,z_o} = o, \left(\frac{\partial p}{\partial t}\right)_{x_o,y_o,z_o} = o, \left(\frac{\partial e}{\partial t}\right)_{x_o,y_o,z_o} = o$$

Where $(x_{o,}y_o, z_o)$ is a fixed point in fluid field

Unsteady flow is that type of flow, is that type of flow in which the velocity, pressure and density at a point changes with respect to time. For unsteady flow,

$$\left(\frac{\partial v}{\partial t}\right)_{x_o,y_o,z_o} \neq o, \left(\frac{\partial p}{\partial t}\right)_{x_o,y_o,z_o} \neq o \text{ e.t.c}$$

- **Uniform and Non-Uniform flows**

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (i.e. length of direction of the flow). For uniform flow

$$\left(\frac{\partial v}{\partial s}\right)_{t=cons\tan t} = o$$

Where $\partial v =$ change of velocity

$\partial s =$ Length of flow in the direction S

This statement implies that other fluid variables do not change with distance. Thus,

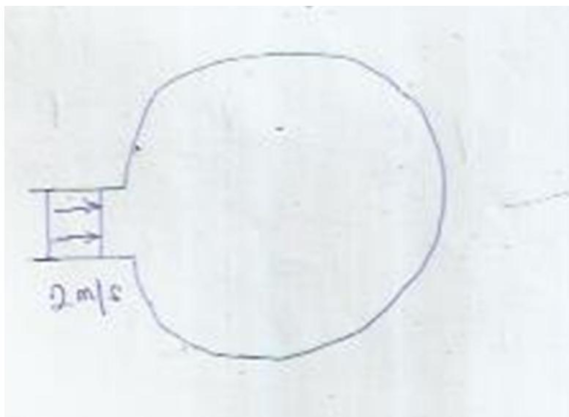
$$\frac{dv}{ds} = o, \frac{\partial p}{\partial s} = o, \frac{\partial \rho}{\partial s} = o$$

Also, in a uniform flow, the velocity (v) is identically same at a

Application of laws governing the motion of bodies to fluid motion

Example 1

A gas flows into a rigid container initially evacuated. Assume that the inflow velocity is uniform at 2mls, as shown below. The tube inlet diameter is 10cm with the volume of the tank equal to 2000 litres. The pressure and temperature in the inlet line are maintained constant at 400kpa and 330K respectively. The gas can be assumed to obey the perfect gas law $P = \rho RT$, with R for the gas equal to 0.30KJ/kg.K. Assume the tank to be non-insulated so that the temperature of the gas in the tank remains constant at a room temperature of 300K. Determine the time required for the pressure in the tank to reach 300kPa.



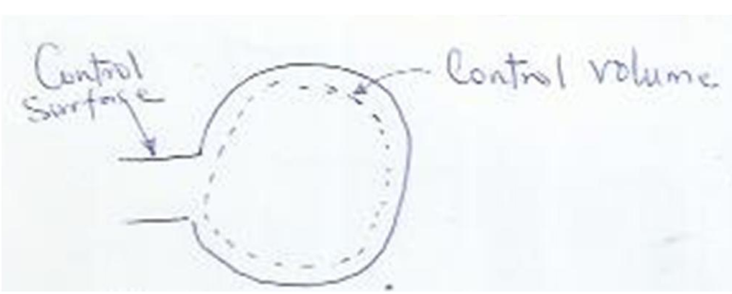
Solution

Flow into the chosen control volume occurs only at the inlet pipe.

From

$$\left.\frac{dM}{dt}\right|_{system} = \left.\frac{\partial x}{\partial t}\right|_{control\,volume} + \int_{cs(inlet\ pipe)} \rho V_n dA = 0$$

$$0 = \left.\frac{\partial x}{\partial t}\right|_{cv} + \int_{(inlet\ pipe)} \rho V_n dA$$



V_n equal 2mls (Efflux is positive) and

$$\rho = \frac{P}{RT} = \frac{400}{0.30\,x\,330} \quad \frac{kN/m^2}{kN.m/kg.K \times K}$$
$$= 4.04kg/m^3$$

Therefore, the second term of the side becomes (4.04) 2dA since density and velocity are constant across the inlet area, they can be taken outside the integral, yielding.

$$(-4.04)(2m/s)x\,\pi/4\,(0.10)^2 = -0.0635kg/s$$
$$therefore,\ \partial m/\partial t = 0.0635kg/s$$
$$\frac{\partial M}{\partial t} = \frac{V\partial\rho}{\partial t} = \frac{V}{Rt}\,x\,\frac{\partial p}{\partial t} \qquad Note: \rho = \frac{p}{RT}$$

Since pressure is only a function of time, we can write the total derivation

$$\frac{V}{RT} \times \frac{dp}{dt} = 0.0635kg/s$$

Integrating, we have

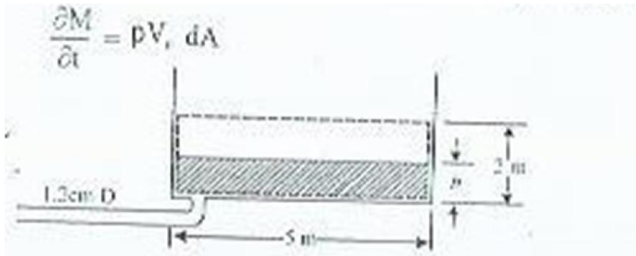
$$\frac{V}{RT}\int_0^{300kpa} dp = 0.0635\int_0^t dt$$

Substituting

$$\frac{2(300)}{0.30\,x\,300}\ \frac{kN/m^2 \times m^3}{kNm/kg.K.K} = 0.0635tKg$$
$$6.7 = 0.0635t\,Kg$$
$$t \approx 105kg$$

Ex 2.

A circular Swimming Pool is 5m in diameter. It is to be filled to a uniform depth of 2m by means of a 1.2cm diameter hose, as shown below. The velocity of the water in the hose is 3mls. Determine the time required to fill the pool in hours.



Solution

Select the control volume to include the entire volume to be filled, as indicate in the diagram

$$0 = \frac{\partial M}{\partial t}\Big|_{cv} + \int_{cv} \ell V_n dA$$

Since the density of water is relatively insensitive to pressure, we have

$$\frac{\partial M}{\partial t} = \ell V_n\, dA$$

The mass of water in the pool at any given time can be expressed as

$$M = \ell \tfrac{\pi}{4} 5^2 h$$

Therefore $\frac{\partial M}{\partial t} = \ell \frac{\pi}{4}\left(5^2\right)\frac{dh}{dt} = \ell V_n A$

Or

$$\frac{dh}{dt} = \frac{3\ x\ \pi/4\ x\ 0.01^2\ x\ 2}{\pi/4\ x\ 5^2} = 17.28\, x\ 10^{-6}\ m/s$$

Integrating, we have

$$\int_0^{2m} dh = 0.00001728 \int_0^t dt$$

$$t = \frac{2}{2\ x\ 10^{-5}}$$

$$= 1157415$$
$$= 32.15 hrs.$$

VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

Velocity Potential function: It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by $\phi(phi)$. Mathematically, the velocity potential is defined as $\phi = f(x, y, z)$ for steady flow such that:

$$\mu = \frac{-\partial\phi}{\partial x}, v = \frac{-\partial\phi}{\partial y}, w = \frac{-\partial\phi}{\partial z}$$

u, v, w are the components of velocity in x, y, z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms fo velocity potential function are given by:

$$u_r = \frac{\partial\phi}{\partial r}, u_\theta = 1/r \frac{\partial\phi}{\partial\theta}$$

U_r = Velocity component in radial (r) direction

U_θ= Velocity component in tangential direction (θ direction)

The continuity equation for an incompressible steady flow is

$$\frac{-\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substituting the values of u, v and w from equation above , we get

$$\frac{\partial}{\partial x}\left(\frac{-\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{-\partial\phi}{\partial z}\right) = 0$$

Or $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \{Laplace equation\}$

For two-dimension case the equation reduce to

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Properties of the potential function

The rotational components* are given by

$$w_z \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$w_y = \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$w_x = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

Substituting the values, of u, v and w from

$$u = \frac{-\partial v}{\partial x}, v = \frac{-\partial \phi}{\partial y}, w = \frac{-\partial \phi}{\partial z}$$

$$w_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{-\partial \phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]=\frac{1}{2}\left[\frac{\partial_2 \phi}{\partial x \partial y}+\frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$w_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(\frac{-\partial \phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]=\frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x}+\frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$w_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{-\partial \phi}{\partial z}\right)\right]-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)=\frac{1}{2}\left[\frac{-\partial^2 \phi}{\partial y \partial z}+\frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$

then $=\frac{\partial^2 \phi}{\partial z \partial x}=\frac{\partial^2 \phi}{\partial x \partial z}; etc$

$$\therefore w_x = w_y = w_x = 0$$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are:

1. If velocity potential ϕ exists, the flow should be irrotational
2. If velocity potential ϕ (satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function

It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (Psi) and defined only for two dimensional flow.

Mathematically, for steady flow it is defined as ψ =f(x,y) such that

$$\frac{\partial\psi}{\partial x} = v \, and \, \frac{\partial\psi}{\partial y =} - u$$

The velocity component in cylindrical polar co-ordinates in terms of stream function are given as

$$U_r = \frac{1}{r}\frac{\partial\psi}{\partial\phi} and \, U_\theta = -\frac{\partial\psi}{\partial r}$$

U_r=radial velocity and U_θ= tangential velocity

The continuity equation for two-dimensional flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the value of u and v in

$$\frac{\partial\psi}{\partial x} = v, \frac{\partial\psi}{\partial y} = -u \, we \, have$$

$$\frac{\partial}{\partial x}\left(\frac{-\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial\psi}{\partial x}\right) = 0 \, or \frac{-\partial^2\psi}{\partial x\partial y} + \frac{-\partial^2\psi}{\partial x\partial y} = 0$$

Hence existence 0 φmeans a possible case of fluid flow. The fluid may be rotational or irrotational

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{-\partial z}{\partial x} - \frac{-\partial u}{\partial y}\right)$ Substituting the values of u and v above equation, we have

$$\omega_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - \left(\frac{-\partial\psi}{\partial x}\right) - \frac{-\partial}{\partial y}\left(\frac{-\partial\psi}{\partial y}\right)\right)$$

$$= \frac{1}{2}\left[\frac{-\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]$$

For irrotational flow, ω_z=0. Hence above equation becomes $\frac{-\partial^2\psi}{\partial x^2}+\frac{-\partial^2\psi}{\partial y^2}=0$

Which is Laplace equation for ψ

The properties of stream function ψ are:

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational
2. If stream function (ψ) satisfies Laplace equation, it is a possible case of an irrotational flow

Equipotential line: A line along which the velocity potential ϕ is constant is called equipotential line.

For equipotential line ϕ = Constant

$$\partial\psi = 0$$

$$\phi = f(x, y)\, for\, steadly\, flow$$

But
$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$
$$= -udx - vdy$$
$$= -(udx + vdy)$$

For equipotential line, $d\phi = 0$

$$-(udx + vdy) = 0\, or\, udx + vdy = 0$$
$$\therefore \frac{\partial y}{\partial x} = \frac{-u}{v}\, but \frac{\partial y}{\partial x} = slope\, of\, equipotential\, line$$

Line of constant stream function

ψ =constant

$d\psi$=0

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = +vdx - udy`$$

For a line of constant stream function

$$= d\psi = 0 \, or \, vdx - udy = 0$$

$$\frac{\partial y}{\partial x} = \frac{v}{u} but \frac{\partial y}{\partial x} is \, slope \, of \, stream \, line$$

From equation $\frac{\partial y}{\partial x} = \frac{-u}{v}$ (slope of equipotential line) and $\frac{\partial y}{\partial x} = \frac{u}{v}$ (slope of stream line). It is clear that the product of slope of equipotential line and the slope of stream line at the point of intersection is equal to-1. Thus the equipotential lines are orthogonal to the stream lines at all points of intersection.

Relationship between stream Function and velocity potential function.

From u= - $\frac{\partial \phi}{\partial x} = and \, v = \frac{\partial \phi}{\partial y} and$

$$u = -\frac{\partial \psi}{\partial y} and \, v = \frac{\partial \psi}{\partial x}$$

Thus we have $u = \frac{-\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial y} and$

$$v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

Hence $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} and \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}$

Example 1

The velocity potential function is given by an expression $\phi = \frac{-xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$

(i) Find the velocity components in x and y directions
(ii) Show that ϕ represents a possible case of flow

Given $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3 y}{3} + y^2$

The partial derivations of ϕ w.r.t to x and y are:

$$\frac{\partial \phi}{\partial x} = \frac{-y^3}{3} - 2x + \frac{3x^2 y}{3}$$

$$and \frac{\partial \phi}{\partial y} = \frac{-3xy^2}{3} + \frac{x^3}{3} + 2y$$

(1) The velocity components u and v are given

$$u = \frac{-\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}, w = \frac{-\partial \phi}{\partial z}$$

$$\Rightarrow u = \frac{-\partial \phi}{\partial x} = -\left[\frac{y^3}{3} - 2x + \frac{3x^2 y}{3}\right]$$

$$= \frac{y^3}{3} + 2x \frac{-3x^2 y}{3}$$

$$= \frac{y^3}{3} + 2x - x^2 y$$

$$v = \frac{\partial \phi}{\partial y} = -\left\{\frac{-3xy^2}{3} + \frac{x^3}{3} + 2y\right\}$$

$$= xy^2 - x^3/3 - 2y$$

(iii) The given value of ϕ , will represent a possible case of flow if it satisfies the Laplace equation i.e. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

But $\frac{\partial \phi}{\partial x}$ is solved to be $\frac{-y^3}{3} - 2x + x^2 y$

And $\frac{\partial \phi}{\partial y}$ is solved to be $-xy^2 + \frac{x^3}{3} + 2y$

$$\therefore if\ \frac{\partial\phi}{\partial x} = \frac{-y^3}{3} - 2x + x^2 y$$

$$\frac{\partial^2\phi}{\partial x^2} = -2 + 2xy$$

$$and\ if\ \frac{\partial\phi}{\partial y} = -xy^2 + \frac{x^3}{3} + 2y$$

$$\frac{\partial^2\phi}{\partial y^2} = -2xy + 2$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (-2 + 2xy) + (-2xy + 2)$$

$$= -2 + 2xy - 2xy + 2$$

$$= 0$$

:.Laplace equation is satisfied and hence represent a possible case of flow.

Example 2:

The velocity potential function is given by $\phi = 5(x^2 - y^2)$ Calculate the velocity components at the point (4,5)

Solution

$$\phi = 5(x^2 - y^2)$$

$$\frac{\partial\phi}{\partial x} = 10x\ and\ \frac{\partial\phi}{\partial y} = -10y$$

By velocity components u and V are given as: $u = -\frac{\partial\phi}{\partial x} = -10x\ and\ v = -\frac{\partial\phi}{\partial y} = 10y$

The velocity components at the point (4,5) i.e., at x=4, y=5

U = -10 x 4 = -40 unit

V = 10 x 5 = 40 unit

Problem 3

A stream function is given by $\psi = 5x - 6y$ calculate the velocity components and also magnitude and direction of the resultant velocity at any point.

$\psi = 5x - 6y$

$\frac{\partial \psi}{\partial x} = 5\, and \frac{\partial \psi}{\partial y} = -6$

But the velocity components u and v in terms of stream function are given as

$$u = -\frac{\partial \psi}{\partial y} andv = +\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial \psi}{\partial x} = 5units / \sec$$

$$and$$

$$u = -\frac{\partial \psi}{\partial y} = 6uits / \sec$$

Resultant Velocity = $= \sqrt{u^2 + u^2}$

$$= \sqrt{6^2 + 5^2}$$

$$= \sqrt{61}$$

$$= 7.91 unit / \sec$$

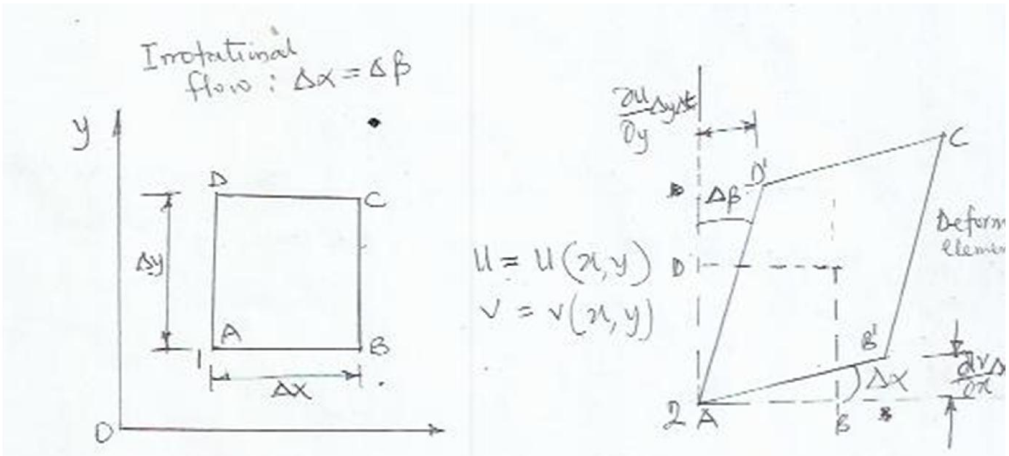
Direction is given by, $\tan = \theta = \frac{v}{u}$

$$\tan\theta = {}^5\!/_6 = 0.8333$$

$$\theta = \tan^{-1}(0.8333) = 39^0\ 48^1$$

Irrotational flow, Vorticity and Circulation

Irrotational flow may be described as flow in which each element of the moving fluid suffers no net rotation from one instant to the next with respect to a given frame of reference. In flow along a curved path fluid elements will deform. If the axes of the element rotate equally towards or away from each other, then the flow will be irrotational. This means that as long as the algebraic average rotation is zero, the flow is irrotational.



Consider a rectangular fluid element of side dx and dy. Under the action of velocities acting on it let- it undergo deformation as shown above in a time Δ t

Δ x = Angular velocity of element AB = $\frac{dv}{dx}$

BB^1 = The Displacement $\frac{dv}{dx}$ Δ x Δ t

Δ B = Angular Velocity of element AD = $\frac{du}{dy}$

Δ D^1 = The displacement $\frac{du}{dy}$ Δ y Δ t

An element is shown moving from point 1 to point 2 along a curved path in the flow field. At 1 the unreformed elements is shown. As it moves to location2, the element is deformed. The angle of rotation of x axis is given by $\left(dv/_{dx}\right).\ \Delta x.\Delta t$. The angle of rotation of y axis is given by $\left(du/_{dy}\right).\ \Delta y.\Delta t$ It is assumed that $\Delta x.\Delta y$ i.e. the angle of rotation toward each other or away from each other should be equal.

The condition to be satisfied for irrotational flow is

$$\frac{dv}{dx} = \frac{du}{dy}.\ or\ \frac{dv}{dx} - \frac{du}{dy} = 0$$

In case there is rotation, then the rotation is given by (with respect to the Z axis in the case of two dimensional flow along x and y)

$$W_z = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy} \right)$$

and

$W_z = 0$ for irrotational flow.

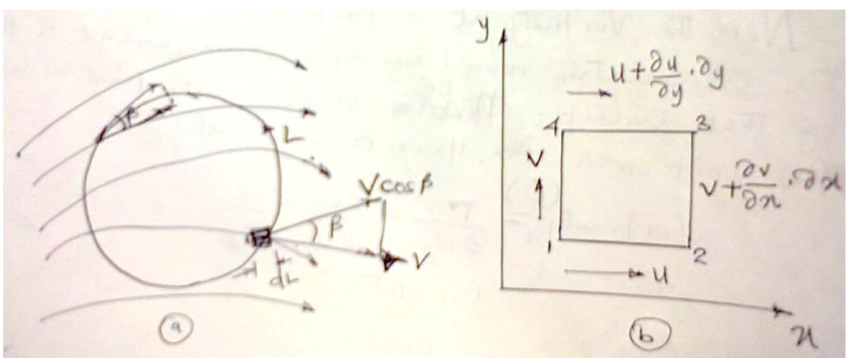
Concept of circulation and Vorticity

Considering a closed path in a flow field as shown below, circulation is defined as the line integral of velocity about this closed path. OR The circulation contained within a closed contour in a body of fluid is defined as the integral around the contour of the component of the velocity vector that is locally tangent to the contour. That is, the Circulation Γ is defined as

$$\Gamma = \oint u ds = \oint u \cos \beta dL$$

Where dl is the length on the closed curve, U is the velocity at the location and β is the angle between the velocity vector and the length DL.

The closed path may cut across several stream lines and at each point the direction of the velocity is obtained from the stream line, as its tangent at that point.



As an example of an elementary circuit arising from the subdivision of a larger one, we consider the elementary rectangle, $\partial x \times \partial y$ in size of the figure (b) above. The velocities along the sides have the directions and average values shown. Starting at the lower left-hand corner, we may add together the products of velocity and distance along each side, remembering that circulation is considered positive anticlockwise.

Consider the element 1234 in the figure (b) above, starting at 1 and proceeding counter clockwise,

$$\text{Circulation } \Gamma = u\partial x + \left(v + \frac{\partial v}{\partial x}\partial x\right)\partial y - \left(u + \frac{\partial u}{\partial y}\partial y\right)\partial x - v\partial y$$

$$= \frac{\partial v}{\partial x}\partial x\partial y - \frac{\partial u}{\partial y}\partial y\partial x$$

OR $$= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\partial x\partial y$$

Now, the vorticity at a point is defined as the ratio of the circulation round an infinitesimal circuit to the area of that circuit . OR Vorticity is defined as circulation per unit area. Here area $\partial x\partial y$, so

$$Vorticity(\xi) = \frac{d\Gamma}{dxdy} = \frac{dv}{dx} - \frac{du}{dy}$$

For irrotational flow vorticity and circulation are both zero. In polar coordinates.

$$Vorticity = \frac{dV_\theta}{dr} - \frac{1}{r}\frac{dV_r}{d_\theta} + \frac{V_\theta}{r}$$

Consider alternatively a small circular circuit of radius r ,

$$\Gamma = \oint uds = \oint \omega r r d\theta = r^2 \oint \omega d\theta = r^2\, \bar{\omega}\, 2\pi$$

Where $\bar{\omega}$ is the mean value of the angular velocity ω about the centre for all particles on the circle.

$$Vorticity(\xi) = \frac{d\Gamma}{dxdy} = \frac{r^2 \varpi 2\Pi}{\Pi r^2} = 2\varpi$$

That is, the vorticity at a point is twice the mean angular velocity of particles at that point. If the vorticity is zero at all points in a region of a flow (except certain special points, called singular points, where a velocity or the acceleration is theoretically zero or infinite) then the flow in that region is said to be irrotational. Flow in regions where the vorticity is other than zero is said to be irrotational.

Assignment

1. Show that the two-dimensional flow described (in metre per second units) by the equation $\psi = x + 2x^2 - 2y^2$ is irrotational. What is the velocity potential of the flow? If the density of the fluid is $1.12 kg/m^3$ and the piezometric pressure at the point (1, -2) is 4.8kPa. what is the piezometric pressure at the point (9,6)?

2. The stream function for a flow is given by ψ = xy. Is the flow irrotational ? Determine (i) u,v (ii) the vorticity and (iii) circulation.

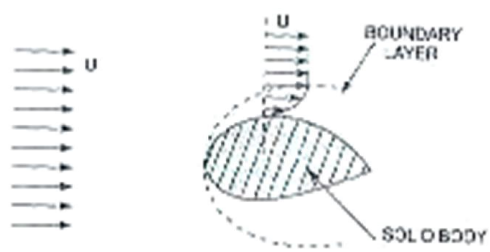
3. Given that $u = x^2 - y^2$ and $v = -2xy$, determine the stream function and potential function for the flow.

BOUNDARY LAYER CONCEPT IN THE STUDY OF FLUID FLOW

When fluids flow over surfaces, the molecules near the surface are brought to rest due to the viscosity of the fluid. The adjacent layers are also slow down, but to a lower and lower extent. This slowing down is found limited to a thin layer near the surface. The fluid beyond this layer is not affected by the presence of the surface. The fluid layer near the surface in which there is a general slowing down is defined as boundary layer. The velocity of flow in this layer increases from zero at the surface to free stream velocity at the edge of the boundary layer.

When a real fluid flow past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. The theory dealing with boundary layer flows is called boundary layer theory.

According to the B.L. theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown below

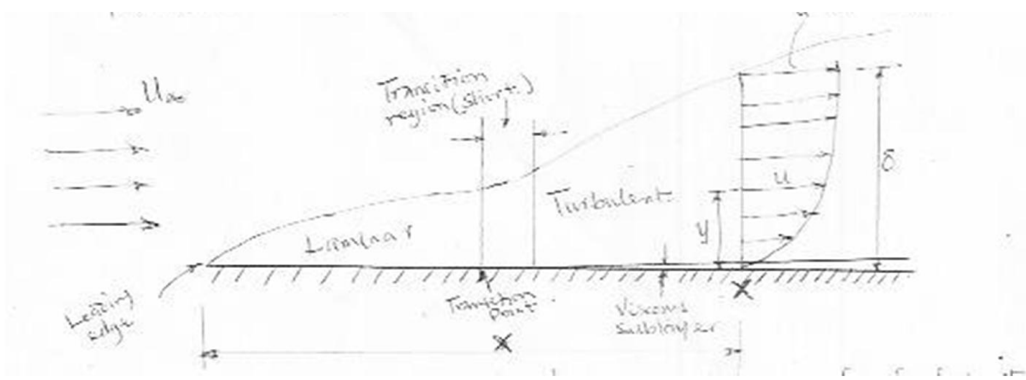


Description of the Boundary Layer

The simplest boundary layer to study is that formed in the flow along one side of a thin, smooth, flat plate parallel to the direction of the oncoming fluid. No other solid surface is near, and the pressure of the fluid is uniform. If the fluid were inviscid no velocity gradient would, in this instance, arise. The velocity gradients in a real fluid are therefore entirely due to viscous action near the surface.

The fluid, originally having velocity U_∞ in the direction of plate, is retarded in the neighbourhood of the surface, and the boundary layer begins at the leading edge of the plate. As more and more of the fluid is slowed down, the thickness of the layer increases. The fluid in contact with the plate surface has zero velocity, ‘no slip’ and a velocity gradient exists between the fluid in the free stream and the plate surface.

The flow in the first part of the boundary layer (close to the leading edge of the plate) is entirely laminar. With increasing thickness, however, the laminar layer becomes unstable, and the motion within it becomes disturbed. The irregularities of the flow develop into turbulence, and the thickness of the layer increases more rapidly. The changes from laminar to turbulent flow take place over a short length known as the transition region.



Graph of velocity u against distance y from surface at point X

Reynolds’ Number Concept

If the Reynolds number locally were based on the distance from the leading edge of the plate, then it will be appreciated that, initially, the value is low, so that the fluid flow close to the wall may be categorized as laminar. However, as the distance from the leading edge increases, so does the Reynolds number until a point is reached where the flow regime becomes turbulent.

For smooth, polished plates the transition may be delayed until Re equals 500000. However, for rough plates or for turbulent approach flows transition may occur at much lower values. Again, the transition does not occur in practice at one well-defined point but, rather, a transition zone is established between the two flow regimes.

The figure above also depicts the distribution of shear stress along the plate in the flow direction. At the leading edge, the velocity gradient is large, resulting in a high shear stress. However, as the laminar region progresses, so the velocity gradient and shear stress decrease with thickening of the boundary layer. Following transition the velocity gradient again increases and the shear stress rises.

Theoretically, for an infinite plate, the boundary layer goes on thickening indefinitely. However, in practice, the growth is curtailed by other surfaces in the vicinity.

Factors affecting transition from Laminar to Turbulent flow Regimes

As mentioned earlier, the transition from laminar to turbulent boundary layer condition may be considered as Reynolds number dependent, $\mathrm{Re}_x = {}^{\rho U_s x}\!/_{\mu} = \frac{\mu x}{\nu}$ and a figure of 5 x 10^5 is often quoted.

However, this figure may be considerably reduced if the surface is rough. For Re <10^5, the laminar layer is stable; however, at Re near 2 x 10^5 it is difficult to prevent transition.

The presence of a pressure gradient ${}^{dp}\!/_{dx}$ can also be a major factor. Generally, if ${}^{dp}\!/_{dx}$ is positive, then transition Reynolds number is reduced, a negative ${}^{dp}\!/_{dx}$ increasing transition Reynolds number.

Boundary Layer thickness (σ)

The velocity within the boundary layer increases from zero at the boundary surface to the velocity of the main stream asymptotically. Therefore the thickness of the boundary layer is arbitrarily defined as that distance from the boundary in which the velocity reaches 99 per cent of the velocity of the velocity of the free stream ($u = 0.99U_\infty$). It is denoted by the symbol σ. This definition however gives an approximate value of the boundary layer thickness and hence σ is generally termed as nominal thickness of the boundary layer.

The boundary layer thickness for greater accuracy is defined as in terms of certain mathematical expression which are the measure of the boundary layer on the flow. The commonly adopted definitions of the boundary layer thickness are:

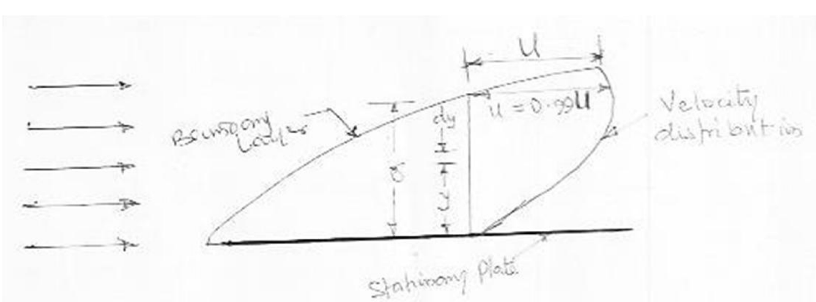
1. Displacement thickens (σ^*)
2. Momentum thickness (θ)
3. Energy thickness (∂_c)

- **Displacement thickness (σ^*)**

The displacement thickness can be defined as the distance measured perpendicular to the boundary by which the main/free stream is displaced on account of formation boundary layer.

Or

It is an additional "Wall thickness" that would have to be added to compensate for the reduction in flow rate on account of boundary layer formation".



Displacement thickness

Let fluid of density ℓ flow past a stationary plate with velocity U as shown above. Consider an elementary strip of thickness dry at a distance y from the plate.

Assumed unit width, the mass flow per second through the elementary strip

$$= \ell u dy ----------------(i)$$

Mass of flow per second through the elementary strip (unit width) if the plate were not there

$$= \ell u dy ----------------(ii)$$

Reduce the mass flow rate through the elementary strip

$$= \ell u dy - \ell u dy$$
$$= \ell (u - u) dy$$

Total momentum of mass flow rate due to introduction of plate

$$= \int_0^\delta \rho (U - u) dy -----------------(iii)$$

(If the fluid is incompressible)

Let the plate is displaced by a distance σ^* and velocity of flow for the distance σ^* is equal to the main/free stream velocity (i.e. U). Then, loss of the mass of the fluid/sec. flowing through the distance σ^*.

$$= \rho U \sigma^* -----------(iv)$$

Equating eqns. (iii) and (iv) we get

$$= \rho U \sigma^* = \int_0^\sigma \rho (U - u) dy$$

or

$$\sigma^* = \int_0^\sigma \left(1 - \frac{u}{U}\right) dy$$

Momentum Thickness (θ)

This is defined as the distance which the total loss of momentum per second be equal to if it were passing a stationary plate. It is denoted by θ.

It may also be defined as the distance, measured perpendicular to the boundary of the solid body by which the boundary should be displaced to compensate for reduction in momentum of the flowing fluid on account of boundary layer formation.

Refer to diagram of displacement thickness above,

Mass of flow per second through the elementary strip = $\rho u dy$

Momentum/Sec. of this fluid inside the boundary layer

$= \rho u dy \times U = \rho u^2 dy$

Momentum/sec. of the same mass of fluid before entering boundary layer = $\rho u U dy$

Loss of Momentum/sec. = $\rho u U dy - \rho u^2 dy = \rho u (U - u) dy$

$\therefore$ Total loss of momentum/sec

$$= \int_0^\delta \rho u (U - u) dy \text{ ---------- } (i)$$

Let θ = Distance by which plate is displaced when the fluid is flowing with a constant velocity U. then loss of momentum/Sec. of fluid flowing through distance θ with a velocity U.

$$= \rho \theta U^2 \text{ ------------ } (ii)$$

Equating eqns. (i) and (ii), we have

$$\rho \theta u^2 = \int_0^\delta \rho u (U - u) dy$$

$$OR$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Energy Thickness (∂_e)

Energy thickness is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in K.E of the flowing fluid on account of boundary layer formation. It is denoted by (∂_e)

Refer to the above displacement thickness diagram,

Mass of flow per second through the elementary strip = $\rho u dy$

K.E of this fluid inside the boundary layer = $\frac{1}{2} m u^2 = \frac{1}{2}(\rho u dy) u^2$

K.E of the same mass of fluid before entering the boundary layer

$$\tfrac{1}{2}(\rho u dy) u^2$$

Loss of K.E. through elementary strip

$$\tfrac{1}{2}(\rho u dy)u^2 - \tfrac{1}{2}(\rho u dy)u^2$$
$$= \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy \text{ ------------------ } (i)$$

$\therefore$ Total loss of K.E of fluid = $\int_0^\delta \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy$

Let ∂_e = Distance by which the plate is displaced to compensate for the reduction in K.E

Then loss of K.E. through ∂_e of fluid flowing with velocity

$$U = \tfrac{1}{2}(\rho U \delta_e)U^2 \text{ ------------ } (ii)$$

Equating eqns (i) and (ii), we have

$$\tfrac{1}{2}(\rho u dy)u^2 = \int_0^\delta \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy$$
$$\delta_e = \frac{1}{U^3}\int_0^\delta u\left(U^2 - u^2\right)dy$$

or

$$\int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy$$

Momentum Equation for Boundary Layer by Von Karman

Von Karman suggested a method based on the momentum equation by the use of which the growth of a boundary layer along a flat plate, the wall shear stress and the drag force could be determined (when the velocity distribution in the boundary layer is known). Starting from the beginning of the plate, the method can be wed for both laminar and turbulent boundary layers.

The figure below shows a fluid flowing over a thin plate (placed at zero incidence) with a free stream velocity equal to U. Consider a small length dx of the plate at a distance x from the leading edge as shown in fig. (a). Consider unit width of plate perpendicular to the direction of flow.

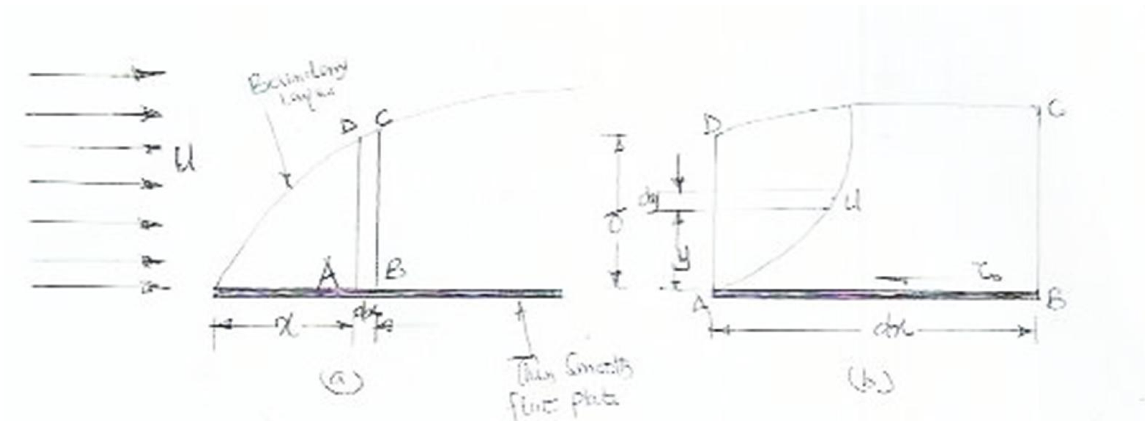


Fig.(a) and (b) Momentum equation for boundary layer by Von Karman

Let ABCD be a small element of a boundary layer (the edge DC represents the outer edge of the boundary layer).

Mass rate of fluid entering through AD

$$=\int_0^\delta \rho u dy+\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx --------(ii)\ \left[\begin{array}{l} i.e.(mass\ through\ AD)+\frac{d}{dx} \\ (mass\ through\ AD)x\,dx \end{array}\right]$$

$\therefore$ Mass rate of fluid entering the control volume through the surface DC

= mass rate of fluid through BC – Mass rate of fluid through AD

$$=\int_0^\delta \rho u dy+\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx-\int_0^\delta \rho u dy -------(iii)$$

$$=\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx ------------------(iv)$$

The fluid is entering through DC with a uniform velocity U.

Momentum rate of fluid entering the control volume of X-direction through AD.

$$\int_0^\delta \rho u^2 dy\ -----------(v)$$

Momentum rate of fluid leaving the Control Volume in X-direction through BC

$$=\int_0^\delta \rho u^2 dy+\frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right]dx ----------(vi)$$

Momentum rate of fluid entering the control volume through DC in X-direction

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u dy\right] dx \times U \quad (\because Velocity = U) ----------(vii)$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u U dy\right] dx ------------(viii)$$

$\therefore$ Rate of change of momentum of Control Volume

= Momentum rate of fluid through BC – Momentum rate of fluid through AD – Momentum of fluid through DC

$$= \int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy\right] dx - \int_0^{\delta} \rho u^2 dy + \frac{d}{dx}\left[\int_0^{\delta} \rho u U dy\right] dx ------(ix)$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \rho u^2 dy - \int_0^{\delta} \rho u U dy \;\; dx\right] ------------$$

$$= \frac{d}{dx}\left[\int_0^{\delta} \left(\rho u^2 dy - \rho u U dy\right)\right] dx$$

$$= \frac{d}{dx}\left[\rho \int_0^{\delta} \left(u^2 - uU\right) dy\right] dx --------------------(x)$$

As per momentum principle, the rate of change of momentum on the control volume BCD must be equal to the total force on the control volume in the same direction. The only external force acting on the control volume is the share force acting on the side AB in the direction B to A (fig. b) above). The value of this force (drag force) is given by,

$$\Delta F_D = \tau_o \times dx$$

Thus the total external force in the direction of the rate of change of momentum $= -\tau_o \times dx$ -------(xi)

Equating equation (x) and (xi), we have

$$-\tau_o \times dx = \rho \frac{d}{dx}\left[\int_{\delta}^{\delta} \left(u^2 - uU\right) dy\right] dx$$

Or

$$\rho \frac{d}{dx}\left[\int_{\delta}^{\delta}\left(u^2 - uU\right)dy\right]$$

$$or, = \rho \frac{d}{dx}\left[\int_{\delta}^{\delta}\left(uU - u^2\right)dy\right]$$

$$= \rho \frac{d}{dx}\left[\int_0^{\delta} U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right]$$

$$= \rho U \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

$$or \quad \frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)\right]dy \text{ ------- } (xiii)$$

But,

$$\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = momentum\ thickness\ \theta$$

$$\therefore \frac{\tau_o}{\rho U^2} = \frac{d\theta}{dx} \text{ -------- } (xvii)$$

This equation is known as von Karman momentum equation for boundary layer flow and it is used to find out the frictional drag on smooth flat plate for both laminar and turbulent boundary layer.

The following boundary conditions must be satisfied for any assumed velocity distribution.

(i) At the surface of the plate $y = 0,\ U = 0, \frac{du}{dy} = finite\ value$

(ii) At the outer edge of boundary layer $y = \delta, u = U,\ y = \delta, \frac{du}{dy} = 0$

The sheer stress, τ_o for a given velocity profile in laminar, transition or turbulent zone is obtained from equations (xii) and (xiii) above. Then drag force on a small distance dx of a plate is given by

$$\Delta F_D = shear\ stress \times area$$

$$= \tau_o \times (B \times dx) = \tau_o \times B \times dx \quad [assuming\ width\ of\ plate\ as\ unity]$$

$$where,\ B = width\ of\ the\ plate$$

$$\therefore Total\ drag\ on\ the\ plate\ of\ length\ L\ one\ side,$$

$$F_D = \int \Delta F_D = \int_0^L \tau_o \times B \times dx$$

- The ratio of the shear stress to the quantity $\frac{1}{2}\ell u^2$ is known as the Local co-efficient of drag" (or co-efficient of skin fraction) and is denoted by C_D^* i.e. $C_D^* = \frac{\tau_o}{\frac{1}{2}\rho u^2}$
- The ratio of the total drag force to the quantity $\frac{1}{2}\ell u^2$ is called 'Average-coefficient of drag' and is denoted by C_D i.e. $C_D^* = \frac{F_D}{\frac{1}{2}\rho A U^2}$

ℓ =Mass density of fluid

A = Area of surface/plate, and

U = free stream velocity

EXAMPLE 1

The velocity distribution in the boundary layer is given by $\frac{u}{U} = \frac{y}{\sigma}$, where u is the velocity y from the plate and u=U at , $y = \delta, \delta$ being boundary layer thickness. Find

i. The displacement thickness
ii. The momentum thickness
iii. The energy thickness and
iv. The value of $\frac{\delta *}{\theta}$

Solution:

Velocity distribution: $\frac{u}{U} = \frac{y}{\sigma}$

(i) The displacement thickness $\delta *$

$$\delta * = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta$$

$$= \left(\delta - \frac{\delta^2}{2\delta} \right) = \delta - \frac{\delta}{2}$$

$$= \frac{\delta}{2}$$

(ii) The momentum thickness

$$\theta = \int_o^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_o^\delta \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)dy$$

$$= \int_o^\delta \left(\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)dy$$

or

$$\theta = \left[\frac{y^2}{2\delta}-\frac{y^3}{3\delta^2}\right]_0^\sigma = \frac{\delta^2}{2\delta}-\frac{\delta^3}{3\delta^2} = \frac{\delta}{2}-\frac{\delta}{3} = \frac{\delta}{6}$$

(iii)

$$\delta_e = \int_o^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

$$= \int_o^\delta \frac{y}{\delta}\left(1-\frac{y^2}{\delta^2}\right)dy = \int_o^\delta \left(\frac{y}{\delta}-\frac{y^3}{\delta^3}\right)dy$$

$$= \left[\frac{y^2}{2\delta}-\frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta}-\frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2}-\frac{\delta}{4}$$

$$= \frac{\delta}{4}$$

(iv) The value of $\frac{\delta *}{\theta}$

$$\frac{\delta *}{\theta} = \frac{\delta/2}{\delta/6}$$

$$= 3.0$$

Example 2

The velocity distribution in the boundary layer is given by $\frac{u}{U} = \frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}$, σ being the boundary layer thickness

Calculate the following

(i) The ratio of displacement thickness to boundary layer thickness $\left(\frac{\delta *}{\delta}\right)$

(ii) The ratio of momentum thickness to boundary layer thickness $\left(\dfrac{\theta}{\delta}\right)$

Solution

Velocity distribution: $\dfrac{u}{U} = \dfrac{3}{2}\dfrac{y}{\sigma} - \dfrac{1}{2}\dfrac{y^2}{\sigma^2}$

(i) $\dfrac{\delta *}{\delta}$:

$$\delta * = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{3}{2}\frac{y}{\sigma} + \frac{1}{2}\frac{y^2}{\sigma^2}\right) dy$$

$$= \left[y - \frac{3}{2} \times \frac{y^2}{3\sigma} + \frac{1}{2} \times \frac{y^3}{3\sigma^2} \right]_0^\delta$$

$$\left[\sigma - \frac{3}{4}\frac{\sigma^2}{\sigma} + \frac{1}{2}\frac{\sigma^2}{3\sigma^2} \right]$$

$$= \left(\sigma - \frac{3}{4}\sigma + \frac{\sigma}{6} \right)$$

$$\sigma * = \frac{5}{12}\sigma$$

$$\therefore \frac{\sigma *}{\sigma} = \frac{5}{12}\sigma.$$

(iii) θ/σ

$$\theta = \int_0^\sigma \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}\right)\left(1-\frac{3}{2}\frac{y}{\sigma} + \frac{1}{2}\frac{y^2}{\sigma^2}\right)dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{9}{4}\frac{y^2}{\sigma^2} + \frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{2}\frac{y^2}{\sigma^2} + \frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$= \int_0^\sigma 3/2\,\frac{y}{\sigma} - \left(\frac{9}{4}\frac{y^2}{\sigma^2} + \frac{1}{2}\frac{y^2}{\sigma^2}\right) + \left(\frac{3}{4}\frac{y^3}{\sigma^3} + \frac{3}{4}\frac{y^3}{\sigma^3}\right) - \frac{1}{4}\frac{y^4}{\sigma^4}\ \Bigg]dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{11}{4}\frac{y^2}{\sigma^2} + \frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$= \left[\frac{3}{2}\frac{y^2}{2\sigma} - \frac{11}{4}\frac{y^3}{3\sigma^2} + \frac{3}{2}\times\frac{y^4}{4\sigma^3} - \frac{1}{4}\times\frac{y^5}{4\sigma^4}\right]_0^\delta$$

$$= \left[\frac{3}{2}\times\frac{y^2}{2\sigma}\times\frac{11}{4}\times\frac{y^3}{3\sigma^2} + \frac{3}{2}\times\frac{\sigma^4}{4\sigma^3} - \frac{1}{4}\times\frac{\sigma^5}{5\sigma^4}\right]_0^\delta$$

$$\theta = \left(\frac{3}{4}\sigma - \frac{11}{12}\sigma + \frac{3}{8}\sigma - \frac{1}{20}\sigma\right) = \frac{19}{120}\sigma$$

$$or\ \frac{\theta}{\sigma} = \frac{19}{120}$$

<u>Assignment</u>

(1) If velocity distribution in laminar boundary layer over a flat plate is given by second order polynomial $U=a + by + cy^2$, determine its form using the necessary boundary conditions

(2) The velocity distribution in the boundary layer is given by $\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}}$, calculate the following

(i) Displacement thickness
(ii) Momentum thickness
(iii) Shape factor
(iv) Energy thickness and
(v) Energy loss due to boundary layer if at a particular section, the boundary layer thickness is 25mm and the free stream velocity is 15m/s. If the discharge through the boundary layer region is $6m^3/s$ per metre width, express this energy loss in terms of metres of head. Take $\ell = 1.2kg/m^3$

(3) In the boundary layer over the face of a high spillway, the velocity distribution was observed to have the following form:

$$\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{0.22}$$

The free stream velocity U at a certain section was observed to be 30m/s and boundary layer thickness of 60mm was estimated from the velocity distribution measured at the section. The discharge passing over the spillway was $6m^3/s$ per metre length of spillway, calculate

i. The displacement thickness
ii. The energy thickness, and
iii. The loss of energy up to the section under consideration.

Laminar Boundary Layer

Let us find out boundary layer thickness (σ), shear stress (τ_o) local co-efficient of drag (C_D) for the following velocity distribution in the boundary layer:

1. $$\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$$

2. $$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\sigma}\right) - \tfrac{1}{2}\left(\frac{y}{\sigma}\right)^3$$

3. $$\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - 2\left(\frac{y}{\sigma}\right)^3 + \left(\frac{y}{\sigma}\right)^4$$

4. $$\frac{u}{U} = Sin\left(\frac{\pi}{2}\frac{y}{\sigma}\right)$$

Case 1: Velocity distribution: $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$ ------- (i)

(i) Boundary layer thickness

We know, $\frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^\sigma \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right]$

Substituting the value of $\frac{u}{U}$, we get

$$\frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\frac{2y}{\sigma} + \frac{y^2}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{4y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^4}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{5y^2}{\sigma^2} + \frac{4y^3}{\sigma^3} - \frac{y^4}{\sigma^4}\right)dy\right]$$

$$= \frac{d}{dx}\left[\frac{2}{2}\,\frac{y^2}{\sigma} - \frac{5}{3}\frac{y^2}{\sigma^2} + \frac{4}{4}\frac{y^4}{\sigma^3} - \frac{1}{5}\frac{y^5}{\sigma^4}\right]_0^{\sigma}$$

$$= \frac{d}{dx}\left[\sigma - \frac{5}{3}\sigma + \sigma\frac{1}{5}\sigma\right] = \frac{d}{dx}\left(\frac{2}{15}\sigma\right)$$

$$\therefore \tau_o = \rho U^2 \times \frac{d}{dx}\left(\frac{2}{15}\sigma\right) = \frac{2}{15}\rho U^2\,\frac{d\delta}{dx} ----------(ii)$$

Also, according to Newton's law of viscosity

$$\tau_o = \mu\left(\frac{dy}{dx}\right)_{y=0} ------------(iii)$$

$$But \quad u = U\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)$$

$$and \quad \frac{du}{dx} = U\left(\frac{2}{\sigma} - \frac{2y}{\sigma^2}\right), U\ being\ cons\tan t$$

$$\therefore \left(\frac{du}{dx}\right)_{y=0} = U\left(\frac{2}{\sigma} - 0\right) = \frac{2U}{8}$$

Substituting this value in (iii), we get

$$\tau_o = \frac{2\mu U}{\sigma} -----------(iv)$$

Equating the values of τ_o given by equations (ii) and iv, we get

$$\frac{2}{15}\rho U^2 \frac{d\sigma}{dx} = \frac{2\mu U}{\sigma}$$

$$or \;\; \sigma.\frac{d\sigma}{dx} = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U^2}$$

$$or \;\; \sigma.d\sigma = \frac{15\mu}{\rho U} dx$$

Integrating both sides, we get

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x + c \quad (where \; C = Cons\tan t \; of \; \text{int}\, egration)$$

$$At \;\; x = 0, \; \delta = 0 \therefore \; C = 0$$

$$\therefore \;\; \frac{\delta^2}{2} = \frac{15\mu}{\rho U}$$

$$or \;\; \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \;\; \times x}} = 5.48 \quad \sqrt{\frac{x^2}{\text{Re}_x}}$$

$$\left(where \; \text{Re}_x = \frac{\rho U x}{\varpi}, \right)$$

$$or \;\; \sigma = 5.48 = \frac{x}{\sqrt{\text{Re}_x}} \text{------}(v)$$

(ii) Shear stress τ_o :

From equation (iv), we have

$$\tau_o = \frac{2\mu U}{\sigma}$$

But $\sigma = 5.48 \frac{x}{\sqrt{\text{Re}_x}}$

$$\therefore \tau_o = \frac{2\mu U}{5.48 \frac{x}{\sqrt{\text{Re}_x}}} = \frac{2\mu U\sqrt{\text{Re}_x}}{5.48x} = 0.365\frac{\mu U}{x}\sqrt{\text{Re}_x} \text{--------------}(vi)$$

(iii) Local Co-efficient of drag, C_D^*

$$\tau_o = \frac{0.365\,\mu U}{x} = \sqrt{\mathrm{Re}_x}$$

$$also\ \tau_o = C_D^* \frac{\rho U^2}{2} ----------(vii)\left(where\ C_D^* = local\ coefficient\ of\ drag\right)$$

Equating the two of τ_o, given by equation (vi) and (vii), we get

$$C_D^* = \frac{0.365\mu U}{x}\sqrt{\mathrm{Re}_x}\ \ or\ \ C_D^* = 0.365 \times 2 \times \frac{\sqrt{\mathrm{Re}_x}}{\frac{\rho U x}{\mu}}$$

$$= \frac{0.73}{\sqrt{\mathrm{Re}_x}}$$

(iv) Co-efficient of drag, C_D:

We know that $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

Where, $F_D = \int_0^L \tau_o \times B \times dx$

$$= \int_0^l 0.365\,\frac{\mu U}{x}\sqrt{\mathrm{Re}_x} \times B \times dx$$

$$= 0.365\int_0^l \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times B \times dx \left(\because \mathrm{Re}_x = \frac{\rho U x}{\mu}\right)$$

$$= 0.365\int_0^l \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times \frac{1}{\sqrt{x}} \times B \times dx$$

$$= 0.365\ \mu U\sqrt{\frac{\rho U}{\mu}} \times B\int_0^l x^{-\frac{1}{2}} \times dx$$

$$= 0.365\ \mu U\sqrt{\frac{\rho U}{\mu}} \times B\left[\frac{x^{-\frac{1}{2}}}{\frac{1}{2}}\right]$$

$$= 0.365 \times 2\mu U B\sqrt{\frac{\rho U}{\mu}} \times B\sqrt{L}$$

$$Or\ F_D = 0.73\ \mu U B\sqrt{\frac{\rho U}{\mu}}$$

$$\therefore C_D = \frac{0.73\mu U B\sqrt{\frac{\rho U}{\mu}}}{\frac{1}{2}\rho A U^2}$$

(Where A – area of plate = L x B, L and B being length and width of the plate respectively)

$$\therefore C_D = \frac{0.73\mu UB\sqrt{\frac{\ell UL}{\mu}}}{\frac{1}{2}\ell \times L \times B \times U^2} = \frac{1.46\mu}{\ell L.U}\sqrt{\frac{\ell UL}{\mu}}$$

$$= \frac{1.46\sqrt{\mu}}{\sqrt{\ell L.U}} = 1.46\sqrt{\frac{\mu}{\ell L.U}} = \frac{1.46}{\sqrt{\mathrm{Re}_l}}$$

CASE 2: Velocity distribution: $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\sigma}\right) - \frac{1}{2}\left(\frac{y}{\sigma}\right)^3$

i. Boundary layer thickness δ :

$$\frac{\tau_o}{\ell U^2} = \frac{d}{dx}\left[\int_0^l \frac{u}{U}\left(1-\frac{u}{U}\right)dy\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Substituting the value of $\frac{u}{U}$, we get

$$\frac{\tau_o}{\ell U^2} = \frac{d}{dx}\left[\int_0^\sigma\left(\frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^3}{\sigma^3}\right)\left(1-\frac{3}{2}\frac{y}{\sigma}+\frac{1}{2}\frac{y^3}{\sigma^3}\right)\right]$$

$$= \frac{d}{dx}\left[\int_0^\sigma\left(\frac{3}{2}\frac{y}{\sigma} - \frac{9}{4}\frac{y^2}{\sigma^2} + \frac{3}{4}\frac{y^4}{\sigma^4} - \frac{1}{2}\frac{y^3}{\sigma^3} + \frac{3}{4}\frac{y^4}{\sigma^4} - \frac{1}{4}\frac{y^6}{\sigma^6}\right)dy\right]$$

$$= \frac{d}{dx}\left[\frac{3}{2}\times\frac{1}{2}\frac{y^2}{\sigma} - \frac{9}{4}\times\frac{1}{3}\frac{y^3}{\sigma^2} + \frac{3}{4}\times\frac{1}{5}\frac{y^5}{\sigma^4} - \frac{1}{2}\times\frac{1}{4}\frac{y^4}{\sigma^3} + \frac{3}{4}\frac{y^5}{\sigma^4}\times\frac{1}{7}\frac{y^7}{\sigma^6}\right]$$

$$\frac{d}{dx}\left[\frac{3}{4}\delta - \frac{3}{4}\delta + \frac{3}{20}\delta - \frac{1}{8}\delta - \frac{1}{28}\delta\right]$$

$$= \frac{39}{280}\frac{d\delta}{dx}$$

$$or\ \tau_o = \ell U^2 \times \frac{39}{280}\frac{d\delta}{dx}$$

$$= \frac{39}{280}\ell U^2\frac{d\delta}{dx} ----------------(ii)$$

Also

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0}$$

But

$$u = U\left[\frac{3}{2}\left(\frac{y}{\sigma}\right) - \frac{1}{2}\left(\frac{y}{\sigma}\right)^3\right]$$

And

$$\frac{du}{dy} = U\left(\frac{3}{2\sigma} - \frac{3}{2}\frac{y^2}{\sigma^3}\right)$$

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu U\left(\frac{3}{2\sigma} - 0\right) = \frac{3\mu U}{2\sigma} \text{ ---------}(iii)$$

Equating the two values of τ_o given by equation (ii) and (iii), we get

$$\frac{39}{280}\ \ell U^2\ \frac{d\delta}{dx} = \frac{3\mu U}{2\delta}$$

$$\delta . d\delta = \frac{3}{2}\mu U \times \frac{280}{39} \times \frac{dx}{\rho U^2}$$

Integrating both sides, we get

$$\delta^2 = \frac{420}{39}\frac{\mu}{\ell U}x + c$$

(Where C=constant of integration)

When x=0, $\delta = 0 \therefore C = 0$

$$\therefore \frac{\delta^2}{2} = \frac{420}{39}\frac{\mu}{\rho u}X$$

Or

$$\delta = \sqrt{\frac{420\ x\ 2}{39}.\frac{\mu}{\rho u}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u} \times \frac{x}{x}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u}}.x$$

$$= \frac{4.64x}{\sqrt{\mathrm{Re}_x}}$$

Shear Stress, τ_o

$$\tau_o = \frac{3\mu U}{2\delta}$$

$$But\ \delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}}$$

$$\therefore \tau_o = \frac{3\mu U}{2 \times \frac{4.64x}{\sqrt{\mathrm{Re}_x}}} = \frac{3}{9.28}\ \frac{\mu U\sqrt{\mathrm{Re}_x}}{x} \text{-------}(v)$$

$$= 0.323\ \frac{\mu U}{x}\sqrt{\mathrm{Re}_x}$$

(iii) Local Coefficient of Drag, C_D^* :

$$\tau_o = 0.323\frac{\mu U}{x}\sqrt{\mathrm{Re}_x}$$

$$Also,$$

$$\tau_o = C_D^*\frac{\ell U^2}{2}$$

$$\therefore\ C_D^*\frac{\ell U^2}{2} = 0.323\frac{\mu U}{x}\sqrt{\mathrm{Re}_x}$$

$$or\ \ C_D^*\ \ \frac{0.646}{\sqrt{\mathrm{Re}_x}}$$

(iv) Co-efficient of drag (C_D):

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

$$where,\ F_D = \int_0^L \tau_o \times B \times dx$$

$$= \int_0^L \frac{\mu U}{x}\mathrm{Re}_x \times B \times dx$$

$$= 0.323\int_0^L \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times B \times dx$$

$$= 0.323\mu U\sqrt{\frac{\rho U x}{\mu}} \times B\int_0^L x^{-\frac{1}{2}}\,dx = 0.323\mu U\sqrt{\frac{\rho U}{\mu}} \times B\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^L$$

$$= 0.323\mu U\sqrt{\frac{\rho U}{\mu}} \times B \int_0^L x^{-\frac{1}{2}}\, dx$$

$$= 0.323\mu U\sqrt{\frac{\rho U}{\mu}} \times B \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^L$$

$$= 0.323 \times 2\mu U\sqrt{\frac{\rho U}{\mu}} \times B \times \sqrt{L}$$

$$or\ F_D = 0.646\mu U\sqrt{\frac{\rho UL}{\mu}} \times B \text{ --------------- }(vii)$$

$$\therefore C_D = \frac{0.646\mu UB\sqrt{\frac{\rho UL}{\mu}}}{\frac{1}{2}\ell \times L \times B \times U^2}$$

$$= 0.646 \times 2 \times \frac{\mu}{\rho UL} \times \sqrt{\frac{\rho UL}{\mu}}$$

$$= \frac{1.292}{\sqrt{\frac{\rho UL}{\mu}}}$$

$$Or,\ C_D = \frac{1.292}{\sqrt{\mathrm{Re}_L}} \left(where,\ \mathrm{Re}_L = \sqrt{\frac{\rho UL}{\mu}}\right) \text{ ---- }(vii)$$

CASE 3: Velocity Distribution:

$$F_D = 0.636\ \mu UB\sqrt{\frac{\ell UL}{\mu}}$$

$$\therefore C_D = \frac{0.686\mu UB\sqrt{\frac{\ell UL}{\mu}}}{\frac{1}{2}\ell AU^2}$$

$$= \frac{0.686\mu UB\sqrt{\frac{\ell UL}{\mu}}}{\frac{1}{2}\ell \times L \times B \times U^2}$$

$$(where\ A = L \times B)$$

$$= 0.686 \times 2 \times \frac{\mu}{\ell UL} \times \sqrt{\frac{\ell UL}{\mu}}$$

$$=1.372\frac{1}{\sqrt{\frac{\ell UL}{\mu}}}$$

$$C_D=\frac{1.372}{\sqrt{\mathrm{Re}_L}}-----------------(xi)$$

CASE 4: Velocity distribution $\frac{u}{U}Sin\left(\frac{\pi}{2}\frac{y}{\sigma}\right)$.

(i) $\delta=\frac{4.795x}{\sqrt{\mathrm{Re}_x}}$

(ii) $\tau_o=0.327\frac{\mu U}{x}\sqrt{\mathrm{Re}_x}$

(iii) $C_D^*=\frac{0.654}{\sqrt{\mathrm{Re}_x}}$

(iv) $C_D=\frac{1.31}{\sqrt{\mathrm{Re}_L}}$

Example 1

Air at atmospheric pressure and at 400K flows over a flat plate with a velocity of 5m/s. The transition from laminar to turbulent flow is assumed to take place at a Reynold number of 5 x 10^5; determine the distance from the leading edge of the plate at which transition occurs.

Solution

At $T_\infty=400K$ and at atmospheric pressure, from tables of properties of air,

$$\rho_a = 0.8826kg/m^3,\ \mu=2.286\times10^{-5}\ kg/m.s$$
$$\nu = 25.90\times 10^{-6}\ m^2/s,\ \Pr = 0.689$$

The transition occurred at a distance L from the leading edge.

$$\mathrm{Re}_L = \frac{U_\infty L}{\nu} = \frac{5\times L}{2.59\times10^{-5}} = 5\times10^5$$
$$L = 2.59m$$

Example 2

Air at atmospheric press and at 350k flows over a flat plate with a velocity of 5m/s. The average drag coefficient Cm over a distance of 2m from the leading edge is 0.0019. Calculate the drag force acting per 1m width of the plate over the distance of 2m from the leading edge.

Solution

From

$$C_D = \frac{F_D}{\frac{1}{2}\ell A U^2}$$

$$F_D = C_D \times \tfrac{1}{2}\ell A U^2$$

$$= WL \ \ C_D \frac{\rho U_\infty^2}{2}$$

At T_∞ of 350k and at atmospheric pressure

$$\rho_a = 0.9980\,Kg/m^3,\ \mu = 2.075 \times 10^{-5}\ \ Kg/m.s.$$

$$\nu = 20.76 \times 10^{-6} m^2/s$$

$$F_D = (1)(2)(0.0019)\frac{(0.9980)(5)^2}{2}$$

$$= 0.0019\ (0.9980)\ (25)$$

$$= 0.0474N$$

$$\text{Re} = \frac{U_\infty L}{\nu} = \frac{5 \times 2}{20.7 \times 10^{-6}} = \frac{10}{20.76} \times 10^6$$

$$= 4.8 \times 10^5$$

The flow is Laminar.

Example 3

Oil with a free stream velocity of 3.0m/s flows over a thin plate 1.25m wide and 2m long. Determine the boundary layer thickness and the shear stress at mid-length and calculate the total, double-sided resistance of the plate ($\rho = 860 kgm^{-3}$, $\nu = 10^{-5} m^2/s,\ \nu = \mu/\rho$)

Solution

Given: $U_s = 3.0m/s,\ width = 1.25m,\ L = 2m.\ \ell = 860kg/m^3$

$$\delta = (U - 0.99U_s)$$
$$U = 0.99 \times 3$$
$$= 2.97m$$

Calculate the Reynolds number at x=1m

$$\mathrm{Re}_x = \frac{U_s x}{\nu} = \frac{3\times 1}{10^{-5}} = 300000 = 3 \times 10^5$$
$$\therefore \mathrm{Re}_x^{\frac{1}{2}} = 547.7$$
$$= 5.48 \times 10^2$$

Note that Re is low enough to allow the laminar boundary layer to survive over the whole plate.

From

$$\tau_o = 0.323 \frac{\mu U}{x} \mathrm{Re}_x{}^{\frac{1}{2}}$$
$$= 0.323 \times 10^{-5} \times 860 \times \frac{3}{1} \times 5.48 \times 10^2$$
$$= 4.57 N/m^2$$
$$\left[Note: \nu = \mu/\rho \, , \mu = \nu\rho \right]$$

The skin friction coefficient (Coefficient of drag) is given by (C_D)

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \, L \times B} = \frac{F_D}{\frac{1}{2}\rho U_3^2 A}$$

$$F_D \; (skin \;\; friction \;\; force) = C_D \times \tfrac{1}{2}\rho U_S^2 \;\; l \times b \;\; (one \;\; side \;\; resis\tan ce)$$

For double sided

$F_D = 2\times\frac{1}{2}\ell U_3^2 \times l\times b\times C_D$

$=2\times\frac{1}{2}\times 860\times 3^2 \times 2\times 1.25\times 1.292\ \mathrm{Re}_l^{\frac{1}{2}}$

Note Re_l *at* $x = 2m = 6\times 10^5$

$\therefore F_D = 2\times\frac{1}{2}\times 860\times 3^2\times 2\times 1.25\times\dfrac{1.292}{\left(6\times 10^5\right)^{\frac{1}{2}}}$

$= 860\times 18\times 1.25\times 1.67\times 10^{-3}$

$= 32.2755N$

$\approx 32.3N$

Example 4

Air at $\dfrac{1}{20}$ atm and at 345K has and $\mu = 2.052\times 10^{-5}\,Kg/ms$. Calculate the prandtl number.

Solution

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho C_P}$$

$$\nu = \frac{2.052\times 10^{-5}}{0.0508} = \frac{0.2052}{508} = 4.0394\times 10^{-4}\,m^2/s$$

$$\alpha = \frac{0.05}{0.0508\ \times 1009} = \frac{5}{5.08\ \times 1009}$$

$$= 9.7547\ \times 10^{-4}\,m^2/s$$

$$\therefore \mathrm{Pr} = \frac{4.0394\times 10^{-4}}{9.7547\times 10^{-4}}$$

$= 0.414$

Turbulent Boundary Layer (TBL)

Turbulent flow

Fluid motion is highly irregular, and is characterized by velocity fluctuations. These fluctuations enhance the transfer of momentum, energy and species, and hence increase surface friction as well as convection transfer rates. Fluid mixing resulting from the fluctuations makes turbulent B.L thickness larger and BL profiles (velocity, temp and conc.) flatter than in laminar flow. In the TBL, 3 different regions may be delineated

(a) Laminar or viscous sublayer – in which transport is dominated by diffusion and the velocity profile is nearly linear.

(b) Buffer layer – adjacent layer to viscous sublayer in which diffusion and turbulent mixing are comparable.
(c) Turbulent zone – transport is dominated by turbulent mixing

The location x_c at which transition begins is determined by a dimensionless grouping of variables called Reynolds numbers

$$\mathrm{Re}_x = \frac{\rho U_\infty X}{\mu}$$

$\mathrm{Re}_{x,c}$ for BL calculation is taken to be 5 x 10^5

For a flow over a flat plate, the value of $\mathrm{Re}_{x,c}$ varies from 1 x 10^5 to 3 x 10^6 depending on surface roughness and the turbulence level of the free stream.

x in the above expression is the characteristic length, the distance measured along the plate.

Turbulent Boundary Layer

As compared to laminar boundary layers, the turbulent boundary layers are thicker. For in a turbulent boundary layer, the velocity distribution is more uniform than in a laminar boundary layer due to intermingling of fluid particles between different layers of the fluid. The velocity distribution in a turbulent boundary layer follows a logarithmic law i.e. u~log y, which can also be represented by a power law of the type.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^n \text{------------} (i)$$

Where, n = $\frac{1}{7}$ (approx..) for Re < 10^7 but > 5 x 10^5

$$\therefore \frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}} \text{------------------} (ii)$$

This is known as one-seventh power law

Let us now find the value of $\delta, \tau_o, C_D^* \; F_D, C_D$ for the velocity distribution given by equation (ii) i.e. $\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}}$

(i) $\sigma = \frac{0.371x}{(\mathrm{Re}_x)^{\frac{1}{5}}}$

(ii) $\tau_o = \frac{\ell U^2}{2} \times \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}}$

(iii) $$C_D^* = \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}}$$

(iv) $$F_D = \frac{\ell U^2}{2} \times \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}} \times B \times L$$

(v) $$C_D = \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}}$$

Note: This is valid for $5 \times 10^5 < \mathrm{Re}_L < 10^7$

For Reynolds no between 10^7 and 10^9, the following relationship suggested by Prandtl and Schlichting hold good

$$C_D = \frac{0.455}{(\log_{10} \mathrm{Re}_L)^{2.58}}$$

Example

Air flows over a smooth flat plate at a velocity of 4.39m/s. The density of air is 1.031kg/m^3 and the kinematic viscosity is 1.34 x 10^{-5}m^2/s. The plate's length is 12.2m in the direction of the flow. Calculate

(a) The boundary layer thickness at 15.24cm and 12.2m respectively from the leading edge.
(b) The drag coefficient C_D, for the plate surface

Solution

At the location x = 15.24cm, the Reynolds number is

$$\mathrm{Re}_x = \frac{Ux}{\nu} = \frac{4.39 \times 15.24 \times 10^{-2}}{1.34 \times 10^{-5}} = 5 \times 10^4$$

and the flow is laminar. The boundary layer thickness is obtained from Blasius solution.

$$\delta = \frac{5x}{\sqrt{\mathrm{Re}\,x}}$$
$$= \frac{5 \times 15.24}{\sqrt{5 \times 104}} = 340.8 \times 10^{-3}\ cm$$

At the location x = 12.2m, the Reynolds number is

$$\mathrm{Re}_x = \frac{4.39 \times 12.2}{1.34 \times 10^{-5}} = 4 \times 10^6$$

And the flow is turbulent. The boundary layer thickness is

$$\sigma = \frac{0.37x}{(\mathrm{Re}_x)^{\frac{1}{5}}} = \frac{0.37 \times 12.2}{(4 \times 10^6)^{\frac{1}{5}}} = 0.216m$$

The drag coefficient C_D can be obtained from

$$C_D = \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}}$$
$$= \frac{0.072}{(5 \times 10^4)^{\frac{1}{5}}}$$
$$= \qquad (for\ laminar\ flow)$$
$$C_D = \frac{0.072}{(4 \times 10^6)^{\frac{1}{5}}}$$
$$= \qquad (for\ Turbulent\ flow)$$

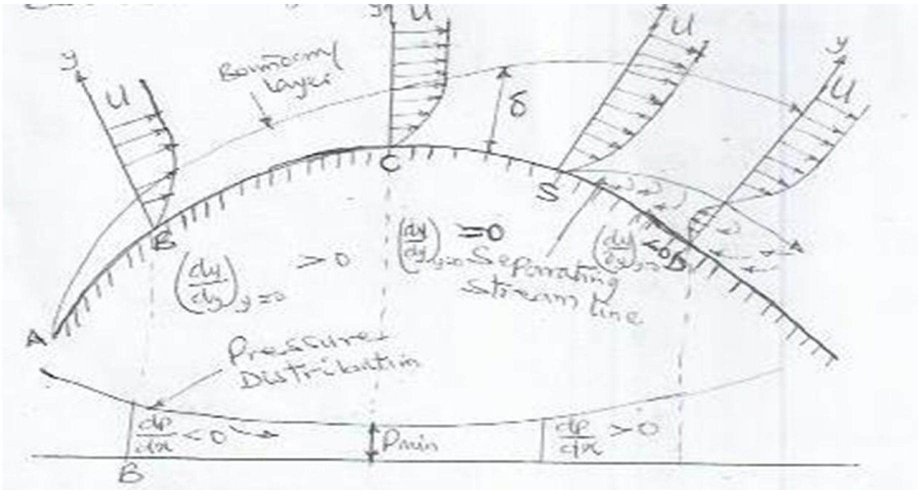
SEPARATION OF BOUNDARY LAYER

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it cannot provide kinetic energy to overcome the resistance offered by the solid body. In other words, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

Effect of Pressure Gradient on Boundary Layer Separation

The effect of pressure gradient $\left(\frac{dp}{dx}\right)$ on boundary layer separation can be explained by considering the flow over a curved surface ABCSD as shown in the figure below. In the region ABC of the curved surface, the area of flow decreases and hence velocity increases. This means that flow get accelerated in this region. Due to the increase of the velocity, the pressure decreases in the direction of the flow and hence pressure gradient $\frac{dp}{dx}$ is negative in this region. As long as $\frac{dp}{dx}<0$, the entire boundary layer moves forward as shown.

Region CSD of the curved: the pressure is minimum at the points C. Along the region CSD of the curved surface, the area of flow increases and hence velocity of flow along the direction of fluid decreases. Due to decrease of velocity, the pressure increases in the direction of flow and hence pressure gradient $\frac{dp}{dx}$ is positive or $\frac{dp}{dx}>0$. Thus is the region CSD, the pressure gradient is positive and velocity of fluid layers along the direction of flow decreases. As earlier mentioned, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Thus the combine effect positive pressure gradient and surface resistance reduces the momentum of the fluid. A stage comes, when the momentum of the fluid is unable to overcome the surface resistance and the boundary layer starts separating from the surface at the point S. Downstream the point S, the flow is taking place in reverse direction and the velocity gradient becomes negative.



Effect of pressure gradient on boundary layer separation

The flow separation depends upon factors such as

(i) The curvature of the surface
(ii) The Reynolds number of flow
(iii) The roughness of the surface

The velocity gradient for a given velocity profile, exhibits the following characteristics for the flow to remain attached, get detached or be on the verge of separation:

1 $\left(\dfrac{du}{dy}\right)_{y=0}$ $is\ +ve$ ------ attached flow (the flow will not separate)

2 $\left(\dfrac{du}{dy}\right)_{y=0}$ $is\ zero$ ------ The flow is on the verge of separation

3 $\left(\dfrac{du}{dy}\right)_{y=0}$ $is\ -ve$ ------ Separated flow

Methods of preventing the Separation of Boundary Layer

The following are some of the methods generally adopted to retard or arrest the flow separation:

1. Streamlining the body shape
2. Tripping the boundary layer from laminar to turbulent by provision of surface roughness
3. Sucking the retarded flow
4. Injecting high velocity fluid in the boundary layer
5. Providing slots near the leading edge
6. Guidance of flow in a confined passage
7. Providing a rotating cylinder near the leading edge
8. Energizing the flow by introducing optimum amount of swirl in the incoming flow

Example

For the following velocity profiles, determine whether the flow is attached or detached or on the verge of separation:

i. $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$ ii. $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^3 + 2\left(\frac{y}{\sigma}\right)^4$

iii. $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^3 + 2\left(\frac{y}{\sigma}\right)^4$

Solution

i. $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2 \; or \; U = 2U\left(\frac{y}{\sigma}\right) - U\left(\frac{y}{\sigma}\right)^2$

Differentiating w.r.t.y the above equation, we get

$$\frac{du}{dy} = 2U\left(\frac{1}{\sigma}\right) - 2U\left(\frac{y}{\sigma}\right) \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} = \frac{2U}{\sigma}$$

$$As \quad \left(\frac{du}{dy}\right)_{y=0} \quad is +ve, the\ given\ flow\ is\ attached$$

ii. $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) + \left(\frac{y}{\sigma}\right)^3 + 2\left(\frac{y}{\sigma}\right)^4$

or $u = -2U\left(\frac{y}{\sigma}\right) + \left(\frac{y}{\sigma}\right)^3 + U\left(\frac{y}{\sigma}\right)^3 + 2U\left(\frac{y}{\sigma}\right)^4$

$$\frac{du}{dy} = 2U\left(\frac{1}{\sigma}\right) - 3U\left(\frac{y}{\sigma}\right)^2 \times \frac{1}{\sigma} + 8U\left(\frac{y}{\sigma}\right)^3 \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} - \frac{2U}{\sigma}$$

$$As \quad \left(\frac{du}{dy}\right)_{y=0} \quad is -ve, the\ given\ flow\ is\ \mathrm{det}ached\ (i.e. the\ flow has\ separated)$$

iii. $\frac{u}{U} = \left(\frac{y}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^3 + 2\left(\frac{y}{\sigma}\right)^4$

$$\text{Or } u = -2U\left(\frac{y}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^3 - 2U\left(\frac{y}{\sigma}\right)^4$$

$$\therefore \frac{du}{dy} = 4U\left(\frac{y}{\sigma}\right) \times \frac{1}{y} + 3U\left(\frac{y}{\sigma}\right) \times \frac{1}{\sigma} - 8U\left(\frac{y}{\sigma}\right)^3 \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} = 0$$

$$As \quad \left(\frac{du}{dy}\right)_{y=0} = 0, \ \textit{the given flow is on the verge of separation}$$

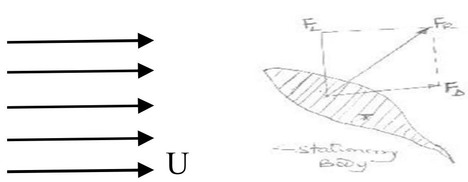
REGIMES OF EXTERNAL FLOW

When a fluid is flowing over a stationary body, a force is exerted by the fluid on the body. Similarly, when a body is moving in a stationary fluid, a force is exerted by the fluid on the body. Also, when both the body and fluid are moving at different velocities, a force is exerted by the fluid on the body. Some of the examples of the fluids flowing over stationary bodies or bodies moving in a stationary fluid are:

(a) Flow of air over buildings,
(b) Flow of water over bridges
(c) Submarines, ships, airplanes and automobiles moving through water and air

Force Exerted by a Flowing fluid on Stationary Bodies

Consider a body held stationary in a real fluid which is flowing at a uniform velocity U as shown in the figure below



Force on a stationary body

The fluid will exert a force on the stationary body. The total force (F_R) exerted by the fluid on the body is perpendicular to the surface of the body. Thus the total force is inclined to the direction of motion.

The total force can be resolved into two components, or in the direction of motion and the other perpendicular to the direction of motion.

DRAG

When a body is immersed in a fluid and is in relative motion with respect to it, the drag is defined as that component of the resultant or total force (F_R) acting on the body which is in the direction of the relative motion. This is denoted by F_D

LIFT

The component of the total or resultant force (F_R) acting in the direction normal or perpendicular to the relative motion is called lift i.e. the force component perpendicular to drag. This component is denoted by F_L. Lift force occurs only when the axis of the body is inclined to the direction of fluid flow. If the axis of the body is parallel to the direction of fluid flow, lift force is zero. In that case only drag force acts. If the fluid is assumed ideal and the body is symmetrical such as a sphere or cylinder, both the drag and lift will be zero.

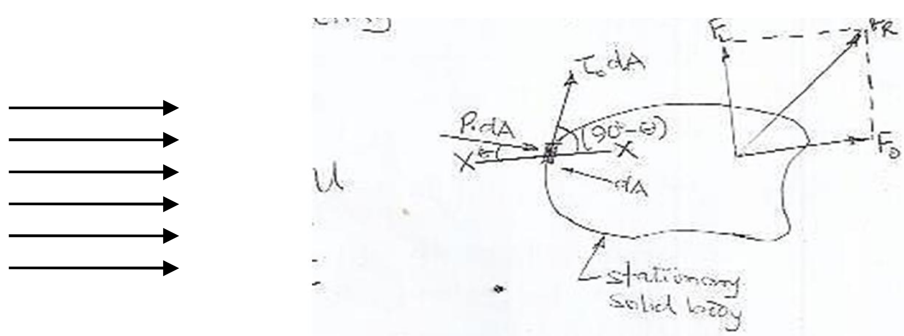
Recall, frictional drag was discussed in connection with the boundary layer theory. It is the force on the body acting in the direction of relative motion due to fluid shear stress at the surface. Thus, in external flow, the immersed body is subjected to frictional drag over its entire surface. Total drag on the body, often called profit drag is therefore made up of two contributions, namely the pressure drag and the skin friction drag. Thus, profile drag = pressure drag + skin frictional drag.

EXPRESSION FOR DRAG AND LIFT

Consider an arbitrary shaped solid body placed in a real fluid, which is flowing with a uniform velocity U in a horizontal direction as shown in the figure below. Consider a small elemental area dA on the surface of the body.

The force acting on the surface area dA are:

1. Pressure force equal to pxdA, acting perpendicular to the surface and
2. Shear force equal to $\tau_o \times dA$, acting along the tangential direction to the surface



Drag and Lift

Let θ = Angle made by pressure force with horizontal direction

(a) Drag force (F_D): The drag force on elemental area = force due to pressure in the direction of fluid motion + force due to shear stress in the direction of fluid motion

$$= PdACos\theta + \tau_o dACos\left(90^\circ - \theta\right) = Pda\ Cos\theta + \tau_o dASin\theta$$

$$\therefore Total\ drag,\ F_D = Summation\ of\ PdACos\theta + Summation\ of\ \tau_o dASin\theta$$

$$= \int PCos\theta dA + \int \tau_o Sin\theta dA ------------(i)$$

The term $\int PCos\theta dA$ is called the pressure drag or form drag while the term $+\int \tau_o Sin\theta\ dA$ is called the friction drag or skin drag or shear drag.

(b) Lift Force (F_L): The lift force on elemental area = Force due to pressure in the direction perpendicular to the direction of motion + Force due to shear stress in the direction perpendicular to the direction of motion

$$= -PdASin\theta + \tau_o\ dASin\theta\left(90^o - \theta\right) = -PdASin\theta + \tau_o\ dACos\theta$$

The negative is taken with pressure force as it is acting in the downward direction while shear force is acting vertically up.

$$\therefore Total\ lift,\ F_L = \int \tau_o dACos\theta - \int pdASin\theta$$

The drag and lift for a body moving in a fluid of density e, at a uniform velocity U are calculated mathematically as

$$F_D = C_D A\frac{\ell U^2}{2}$$

$$F_L = C_L A\frac{\ell U^2}{2}$$

where

C_D= Coefficient of drag

C_L= Coefficient of Lift

A = Area of the body which is the projected area of the body perpendicular to the direction of flow

= largest projected area of the immersed body

Then resultant force on the body, F_R= $\sqrt{F_D^2 + F_L^2}$

Example 1

A flat plate 1.5m x 1.5m moves as 50km/hr in stationary air of density 1.15kg/m^3. If the coefficients of drag and lift are 0.15 and 0.75 respectively. Determine:

(i) The lift force

(ii) The drag force
(iii) The resultant force and
(iv) The power required to keep the plate in motion

Solution

Area of the plate, A = 1.5 x 1.5 = 2.25m^2

Velocity of the plate, U = 50km/hr = $\frac{50 \times 1000}{60 \times 60} = 13.89 m/s$

Density of air, $\ell = 1.15 kg/m^3$

Coefficient of drag, $C_D = 0.15$

Coefficient of lift, $C_L = 0.75$

(i) Lift force (F_L) = $C_L A \frac{\ell U^2}{2}$

$$= 0.75 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} = 187.20N$$

(ii) Drag Force (F_D)= $C_D A \frac{\ell U^2}{2}$

$$= 0.15 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} = 37.44N$$

(iii) Resultant force (F_R) = $\sqrt{F_D^2 + F_L^2} = \sqrt{37.44^2 + 187.20^2}$

$$\sqrt{1400 + 35025}$$

$$= 190.85N$$

(iv) Power Required to keep the plate in motion

$$P = \frac{Force\ in\ the\ direction\ of\ motion \times Velocity}{1000} kW$$

$$= \frac{F_D \times U}{1000} = \frac{37.425 \times 13.89}{1000} kW = 0.519kW$$

Example 2

Find the difference in drag force exerted on a flat plate of size 2m x 2m when the plate is moving at a speed of 4m/s normal to its plane in (i) water (ii) air of density 1.24kg/m^3. Coefficient of drag is given as 1.15.

Solution

Area of plate, A = 2 x 2 = 4m^2

Velocity of Plate, U = 4m/s

Coefficient of drag C_D= 1.15

(i) Drag force when the plate is moving in water

$$F_D = C_D \times A \frac{\rho U^2}{2}$$

(ii) Drag force when the plate is moving in air,

$$F_D = C_D \times L \times \frac{\ell U^2}{2}$$

$$= 1.15 \times 4.0 \times 1.24 \times \frac{4^2}{2} = 45.6N ----------(ii)$$

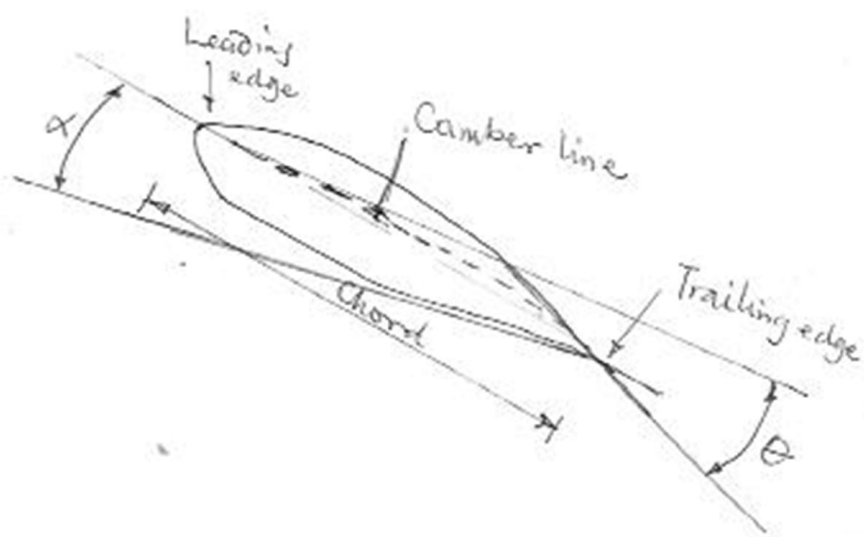
$\therefore$ Difference in drag force = (i) – (ii)

= 36800 – 45.6

= 36754.4N

Flow Past An Infinitely Long Aerofoil.

An aerofoil may be defined as a streamlined body design to produce lift. There are other lift-producing surfaces such as hydrofoils or circular arcs.



Consider the aerofoil or airfoil sketched above, the leading edge is the front or upstream edge facing the direction of flow, while the trailing edge is the rear, or downstream edge. Other important terms relating to airfoil are as follows:

- **Chord line:** This is a straight line joining the centres of curvature of the leading and trailing edges.
- **Chord, C:** The length of chord line between the leading and trailing edges.
- **Camber Line:** The centerline of the airfoil section
- **Camber,** δ **:** The maximum distance between the camber line and the chord line.
- **Deviation** θ **:** The angle between the tangent to camber line at trailing edge and the tangent to camber line at leading edge.
- **Angle of attack** (incidence)**:** The angle between the direction of the relative motion and the chord line
- **Pressure coefficient Cp: (** $p - p_0 / \frac{1}{2}\rho U_o$ **)** where ρ is the local pressure and P_0 is the pressure far upstream of the aerofoil where velocity is Vo.

The primary purpose of an aerofoil is to produce lift when placed in a fluid stream. It will of course, experience drag at the same time. In order to minimize drag, an aerofoil is a streamlined body. A measure of its usefulness as a wing section of an aircraft or as a blade section for a pump or turbine is the ratio of lift to drag. The higher this ratio is, the better the aerofoil, in the sense that it is capable of producing high lift at a small drag penalty. In an aircraft it is the lift on the wing surface which maintains the plane in the air. At the same time, it is the drag which absorbs all the engine power necessary for the craft's forward motion.

The lift/drag ratio

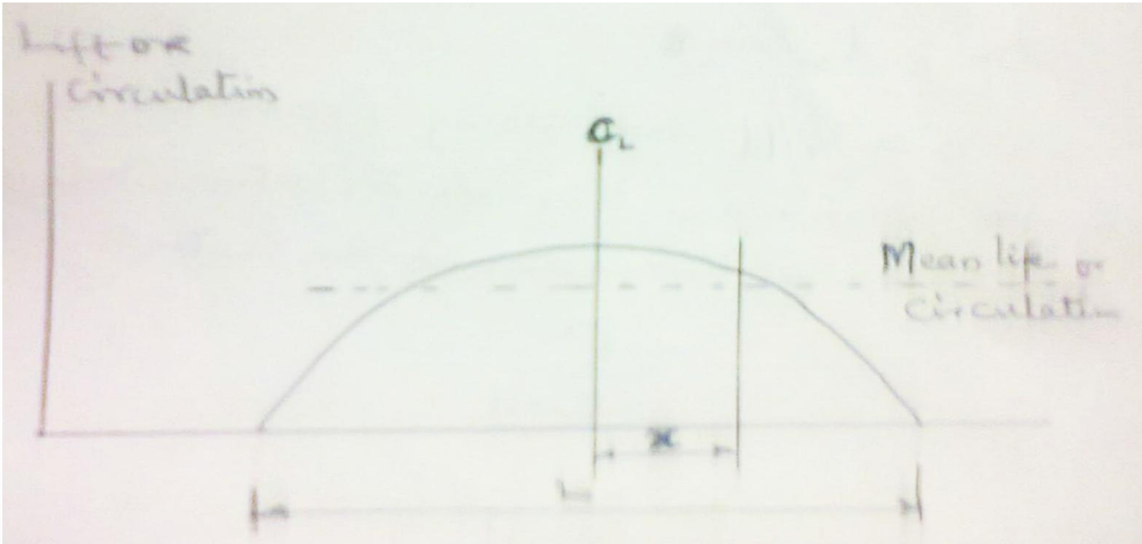
$$\frac{Lift}{Drag} = \frac{\frac{1}{2}\ell C_L U_o^2 A}{\frac{1}{2}\ell C_D U_o^2 A} = \frac{C_L}{C_D}$$

Flow Past an Aerofoil of Finite Length

When an aerofoil is subjected to lift force, the pressure on its underside is greater than that on the top. This pressure difference between the upper and the lower surface causes flow around the tips of the aerofoil from the underside to the upper surface. This end flow affects the rest of the flow pattern in the following manner. The flow on the underside is deflected towards the tips of the aerofoil in order to supply the necessary end flow, whereas the flow at the top of the aerofoil is deflected from the tips towards the centre.

Since there is end flow at the tips, the pressure difference between the top and bottom surfaces of an aerofoil must decrease from a maximum at the middle towards the tips where it is zero.

Consequently, the circulation around the aerofoil finite span must also decrease from its maximum value Γ_a at the centerline towards zero at the tips. The distribution is to be approximated to an ellipse as shown below



Distribution of lift along a wing's span

A further consequence of the tip vortices is that they induce a downward velocity component which is known as downwash velocity $\bar{vi}$. Its presence means that the relative velocity of motion between the fluid and the aerofoil is no longer the free stream velocity Uo but velocity U, deflected from Uo by an angle $\in$ known as the induced angle of incidence. The resulting geometry is shown below. What follows is that, in accordance with the definition of lift, which stipulates that it is perpendicular to the relative direction of motion, the true lift is normal to U. However, since it is more convenient and customary to relate lift and drag to the direction of the free stream relative to the aerofoil, the true lift L_o is resolved into L, the component perpendicular to U_o, and D_i, the component parallel to U_o. This latter component, which is in the same direction as drag is known as induced drag and is added to pressure drag and the skin friction drag to give the total drag on an aerofoil. The expression for induced drag is derived as follows. The true lift per unit length of span is given by

$$L_o = L_o U \tau;$$

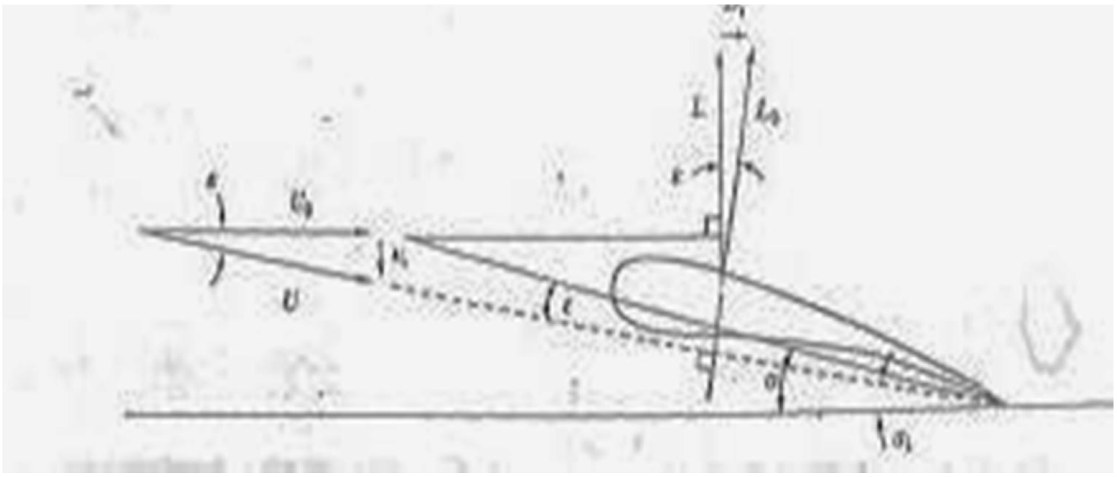
Hence, the induced drag per unit span

$$D_i = L_o Sin \in$$

But $\in = V_i / U$ and, using Prandtl's approximation for elliptical spanwise lift distribution that $V_i = \frac{\Gamma_o}{2bU}$, where

and

$$D_i = \ell U\Gamma\left(\Gamma_o / 2bU\right)$$
$$= \ell\Gamma\left(\Gamma_o \big/ 2b\right)$$



Induced drag

Now, for the elliptic spanwise distribution of Γ ,

$$\Gamma = \Gamma_o\left[1-\left(2x \big/ b\right)^2\right]^{1/2}$$

Where x is the distance from the centerline. Thus, the induced drag for the total span,

$$D_1 = \frac{\ell}{2b}\Gamma_o^2\int_{-b/2}^{+b/2}\left[1-\left(2x \big/ b\right)^2\right]^{1/2} dx = \frac{\ell}{2b}\Gamma_o^2\frac{b\Pi}{4} \qquad (vi)$$
$$= \ell\Pi\Gamma_o^2 / 8$$

is obtained by substitution $2x \big/ b = Sin\,\theta$ But

$$L_0 = \int_{+b/2}^{+b/2} \ell U \Gamma dx = \ell U \Gamma_{+b/2}^{+b/2} \left[1 - \left(2x/b \right)^2 \right]^{1/2} dx$$

$$= \ell U \Gamma_o b \frac{\Pi}{4}$$

From which

$$\Gamma_o = 4L_o / \ell U b \Pi$$

And, substituting into eqn. (vi)

$$D_i = (\ell \Pi / 8) \left(4L_o / \ell U b \Pi \right)^2 = 2L_o^2 / \ell \Pi U^2 b^2$$

However, from similar triangles

$$L_0 / U = L / U_0$$

and, hence,

$$D_i = \left(2 / \ell \Pi b^2 \right) \left(L / U_o \right)^2$$

If the coefficient of induced drag is defined as

$$C_{Di} = \frac{D_i}{\frac{1}{2} \ell U^2 A}$$

And since $C_L = \dfrac{L}{\frac{1}{2} \ell U^2 A}$, , by substitution

$$C_{Di} = \frac{D_i}{L / C_L} = \frac{C_L 2L^2}{L \ell \Pi b^2 U^2} = 2C_L \ \frac{L}{\ell \Pi b^2 U^2}$$

$$= 2C_L \frac{\frac{1}{2} C_L \ell U^2 A}{\ell \Pi b^2 U^2} = C_L^2 \frac{A}{\Pi b^2} = C_L^2 \frac{cb}{\Pi b^2}$$

$$= \frac{C_L^2}{\Pi} \times \frac{c}{b}$$

But

$$\frac{c}{b} = \frac{1}{Aspect\ Ratio},$$

So that

$$C_{Di} = \frac{C_L^2}{\Pi \left(Aspect\ Ratio\right)}$$

This equation shows that a large aspect ratio minimizes the induced drag, as would be expected.

Example 1

A wing of an aircraft of 10m span and 2m mean chord is designed to develop a lift of 45kN at a speed of 400km/h. A 1/20 scale model of the wing section is tested in a wind tunnel at 500m/s and ρ = 5.33 kg/m³. The total drag measured is 400N. Assuming that the wing tunnel data refer to a section of infinite span, calculate the total drag for the full-size wing. Assume an elliptical lift distribution and take air density as 1.2kg/m³.

Solution

Wing area,

$$A = 2 \text{ x } 10 = 20\text{m}^2$$

Coefficient of drag from the model data,

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A} = \frac{\mathbf{400}}{\frac{1}{2} X\ 5.3\ X\ 500^2\ X\ {}^{20}/_{20^2}} = \mathbf{0.012.}$$

For the prototype,

$$U = 400 \text{ km/h} = 111.1\text{m/s}$$

And the lift coefficient,

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 A} = \frac{\mathbf{45000}}{\frac{1}{2} X\ 1.2\ X\ 111.1^2\ X\ 20} = \mathbf{0.304}$$

Now , assuming an elliptical distribution, the coefficient of induced drag,

$$C_{D_0} = C_L^2/\pi(AR) = (0.304)^2/\ \pi\ (\frac{10}{2}) = 0.0059.$$

Hence, the total drag coefficient,

$$C_{D_i} = C_D + C_{D_0} = 0.012 + 0.0059 = 0.0179$$

And the total drag on the wing,

$$D = \frac{1}{2} C_{D_i} \rho U^2 A = \frac{1}{2} \text{ x } 0.0179 \text{ x } 1.2\ (111.1)^2 \text{ x } 20 = 2648.9 \text{ N}$$

Therefore,

$$D = 2.65 \text{ kN.}$$